

VOLUME 42, NUMBER 1, JUNE 2026

ALGEBRAS, GROUPS AND GEOMETRIES Volume 42, Number 1, June 2026

***ALGEBRAS,
GROUPS
AND
GEOMETRIES***

FOUNDED IN 1984. SOME OF THE PAST EDITORS
INCLUDE PROFESSORS: A. A. ANDERSEN,
V. V. DEODHAR, J. R. FAULKNER,
M. GOLDBERG, N. IWAHORI, M. KOECHER,
M. MARCUS, K. M. MCCRIMMON, R. V. MOODY,
R. H. OHEMKE, J. M. OSBORN, J. PATERA,
H. RUND, T. A. SPRINGER, G. S. TSAGAS.

EDITORIAL BOARD

M.R. ADHIKARI
A.K. ARINGAZIN
P. K. BANERJI
A. BAYOUMI
QING-MING CHENG
A. HARMANCY
L. P. HORWITZ
C. P. JOHNSON
S.L. KALLA
N. KAMIYA
J. LISSNER
J. LOHMUS*
R. MIRON
M.R. MOLAEI
P. NOWOSAD
KAR-PING SHUM
SERGEI SILVESTROV
H. M. SRIVASTAVA
G.T. TSAGAS*

* In memoriam

**FOUNDER and
Editor In Chief
R. M. SANTILLI**

H
P

HADRONIC PRESS, INC.

ALGEBRAS, GROUPS AND GEOMETRIES

Established in 1978 by Prof. R. M. Santilli at Harvard University Hadronic Journal and Algebras, Groups and Geometries have been regularly published since 1978 without publication charges and are among the few remaining independent refereed journals.

This Journal publishes advances research papers and Ph. D. theses in any field of mathematics without publication charges.

**For subscription, format and any other information
please visit the website
<http://www.hadronicpress.com>**

**HADRONIC PRESS INC.
35246 U. S. 19 North Suite 215
Palm Harbor, FL 34684, U.S.A.
<http://www.hadronicpress.com>
Email: info@hadronicpress.com
Phone: +1-727-946-0427**

ALGEBRAS, GROUPS AND GEOMETRIES

Founder and Editor in Chief
RUGGERO MARIA SANTILLI

The Institute for Basic Research

35246 U. S. 19 North Suite 215, Palm Harbor, FL 34684, U.S.A.

Email: research@i-b-r.org; TEL: +1-727-688-3992

R.M. ADHIKARI Calcutta Mathematical Society AE-374, Sector-1, Salt Lake City Kolkata, West Bengal 700064

A.K. ARINGAZIN Department of Theoretical Physics, Institute for Basic Research, Eurasian National University, Astana 010008, Kazakhstan
aringazin@mail.kz

P.K. BANERJI Department of Theoretical Physics, Faculty of Science, I.N.V. University, Jofhpur-342 2005, India
bannerjpk@yahoo.com

A. BAYOUMI Alazhar University Faculty of Science Mathematics Department, Naser City 11884 Cairo Egypt
draboubakr@yahoo.com

QING-MING CHENG Department of Mathematics, Faculty of Science Josai University, Sakado, Saitama 350-02 Japan
cheng@math.josai.ac.jp

A. HARMANCY Department of Mathematics, University of Hacettepe, Beytepe Campus, Ankara, Turkey
harmaci@hacettepe.edu.tr

L. P. HORWITZ Department of Physics, Tel Aviv University, Ramat Aviv, Israel
horwitz@taunivm.tau.ac.il

C. P. JOHNSON Department of Mathematics, Mississippi State University P. O. Box MA, Mississippi State, MS 39762
cjohnson@math.msstate.edu

S. L. KALLA Department of Mathematics, Vyas Institutes of Higher Education, Jodhpur 342008 India
shyamkalla@gmail.com

N. KAMIYA University of Aizu Center for Mathematical Sciences, Tsuruga, Ikki-machi Aizu-Wakamatsu City Fukushima, 965-8580 Japan

J. LISSNER, Alumnus

Foukzon Laboratory
Center for Mathematical Sciences
Technion-Israel Institute of Technology
Haifa, 3200003, Israel

L. LOHMUS* Estonia Academy of Sciences, 142 Rita Street, Tartu 202400, Estonia

R. MIRON Faculty of Mathematics University "Al. I. Cuza" 6600 Isai, Romania rmiron@uaic.ro

M.R. MOLAEI Department of Mathematics, University of Kerman, P.O. Box 76135-133 Kerman, Iran
mrmolaei@mail1.uk.ac.ir

P. NOWOSAD Mathematics FFCLRP - USP, Univ. of San Paulo 14040-901 Ribeirao Preto SP, Brazil

KAR-PING SHUM Institute of Mathematics, Yunnan University, Kunming, China
kpshum@ynu.edu.cn

S. SILVESTROV School of Education, Culture and Communication (UKK) Mälardalen University, Box 883, 71610 Västerås, Sweden
sergei.silvestrov@mdh.se

H. M. SRIVASTAVA Department of Mathematics and Statistics, University of Victoria, Victoria, B.C. V8W 3P4, Canada
hmsri@uvvm.uvic.ca

G. F. TSAGAS* Mathematics Department, Aristotle University Thessaloniki 54006, Greece

***In Memoriam**

ALGEBRAS, GROUPS AND GEOMETRIES

VOLUME 42, NUMBER 1, JUNE 2026

THE LORENTZIAN VIVIANI CURVES AT THE LORENTZ-MINKOWSKI SPACE R_1^3 , 1

Tayyibat Taner

Polinas Vocational and Technical Anatolian High School
Ministry of National Education
Manisa 45030, Turkey

Hasan Hüseyin Uğurlu

Department of Secondary School Mathematics and Science Education
Gazi University, Ankara 06560, Turkey

A VARIATIONAL FORMULATION FOR MODELING A BOSE-EINSTEIN CONDENSATION IN AN EULER-BERNOULLIAN CONTEXT, 53

Fabio Silva Botelho

Department of Mathematics
Federal University of Santa Catarina – UFSC
Florianópolis, SC, Brazil

ON PARAMETERIZATION OF SOME PROBLEMS IN NUMBER THEORY, 67

Sergey N. Artekha

Space Research Institute of RAS
117997 Moscow, Russia

STEADY-STATE ENTANGLEMENT CLASS IN FINITE-DIMENSIONAL QUANTUM SYSTEMS CANNOT BE DETERMINED BY FULL NON-ZERO LIOUVILLIAN SPECTRA, 83

Christian G. Barker

Independent Scholar
Cornwall, United Kingdom

MODIFIED QUATERNION AND OCTONION DIVISION ALGEBRAS, 89

S. C. Althoen

Dexter, Michigan

SPECIAL RECTILINEAR CONGRUENCES, 95

Gr. Tsagas, Ch. Christophoridou and As. Synefaki

School of Technology

Department of Mathematics and Physics

Aristotle University of Thessaloniki

Thessaloniki 540 06 - Greece

WALL MONOIDS I, 105

I. G. Rosenberg

Mathématiques et Statistique

Université de Montréal, C. P. 6128

Succ. Centre-ville Montréal, Qué.

Canada H3C 3J7

**INFINITE-DIMENSIONAL HOLOMORPHY WITHOUT CONVEXITY
CONDITION. I: THE LEVI PROBLEM IN NON LOCALLY CONVEX
SPACES, 113**

Aboubakr Bayoumi

King Saud University

College of Science

Department of Mathematics

P. O. Box 2355

Riyadh 11451, Saudi Arabia

THE LORENTZIAN VIVIANI CURVES AT THE
LORENTZ-MINKOWSKI SPACE R_1^3

Tayyibat Taner

Polinas Vocational and Technical Anatolian High School
Ministry of National Education
Manisa 45030, Turkey
tayyibattaner@yahoo.com

Hasan Hüseyin Uğurlu

Department of Secondary School Mathematics and Science Education
Gazi University, Ankara 06560, Turkey
hugurlu@gazi.edu.tr

Received August 8, 2025

Abstract

Spherical curves constitute an important class with widespread application. In Euclidean 3 – space, the intersection of a sphere of radius $2a$ centered at origin and a circular cylinder of radius a centered at $(a,0,0)$ is known as the Viviani's (1622-1703) curve.

In this study, we define four Viviani curves which are intersections of the *Lorentzian sphere* of the Lorentz – Minkowski space R_1^3 with *circular cylinder with index 1*. We find the Frenet formulae, the curvature functions and the Frenet instantaneous rotation vectors for the Frenet frames defined on these curves.

Keywords: Lorentzian Sphere, Circular Cylinder with index 1, Lorentzian Cylinder, Hyperbolic Cylinder, Lorentzian Viviani Curves.

1 Introduction

The most basic concepts in the construction of geometries are curves and surfaces. Considering that surfaces are defined with the help of curves, the importance of curves becomes apparent. The geometry of a plane or space curve is examined with the help of the Frenet framework defined at each point of the curve.

Some special curves defined in space or on a surface; robotics, kinematics, dynamics, computer-aided geometric design, simulation, textile, health, transportation, architecture, etc. Technological developments have increased with the use in different disciplines such as and these curves have played important roles in making our lives easier. For the usage areas of curves, see [1], [2] and [3].

Italian mathematician Vincenzo Viviani proposed the problem of opening four equal windows on a hemispherical dome in 1692. The solution to this problem is given by the intersection of a sphere of radius and a right circular cylinder of radius. This intersection curve on the sphere and cylinder intersects itself at the end points of one diameter of the sphere. This curve, which is located on the sphere surface and resembles the figure-eight curve, is called the Viviani curve [4, 5, 22, 23].

The Viviani curve defined in Euclidean space is a special case of the Clelia curve (spherical spiral) [6]. General clelia curves including the generalized Viviani curve were defined in [5]. Different curves are also defined on the sphere of the Lorentz – Minkowski space R_1^3 [7, 8]. Barros and others have studied general helices in 3-dimensional Lorentzian spaceforms. In [9], Ferrandez et al. stated and proved the Lancret - type theorem for null generalized helices in 3 - dimensional Lorentz - Minkowski space R_1^3 . Furthermore, all helices at the Minkowski 3 – space R_1^3 were classified by Walrave [10]. Uğurlu et al. [11, 12] obtained the differential equations of timelike helices on the surface of the Lorentzian sphere with radius r and gave the parameterizations of these curves. They defined the Frenet and Darboux frames of these curves, derivative formulae, curvature functions, geodesic curvature function and instantaneous rotation vectors. In [21], they also defined

the differential equations of spirals defined on spheres of Minkowski space and expressed generalized parametrization of these special spherical curves. In [13], the differential equations of spirals on oblate and prolate spheroids of space were defined and their parameterizations were given. In [14], all Viviani curves at the Lorentz – Minkowski 3 – space R_1^3 defined and the geometries of the these curves examined with the help of Frenet frames.

In this study, four Viviani curves were defined on the Lorentzian sphere of radius a of the 3 - dimensional Lorentz - Minkowski space R_1^3 . Frenet derivative formulas, curvature and torsion functions were found, and Frenet instantaneous rotation vectors were obtained.

For more information about spherical curves and helical curves, see references [28, 29, 30].

In this study, vectors are shown in bold.

2 Fundamental Concepts

In this section, basic definitions and theorems at the Lorentz-Minkowski 3-space R_1^3 will be given.

The *Lorentz – Minkowski space* R_1^3 is the space R^3 endowed with the metric induced by the Lorentzian scalar product

$$\langle , \rangle : R^3 \times R^3 \rightarrow R$$
$$(\mathbf{x}, \mathbf{y}) \rightarrow \langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3 \quad (2.1)$$

where $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ [15].

Let's point out here that this scalar product, like the Euclidean inner product, is bilinear, symmetric, nondegenerate. But it is indefinite, that is, it is not positive definite.

We say that a nonzero vector $\mathbf{x} \in R_1^3$ is *spacelike* if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ or $\mathbf{x} = 0$, *timelike* if $\langle \mathbf{x}, \mathbf{x} \rangle < 0$, *lightlike(null)* if $\langle \mathbf{x}, \mathbf{x} \rangle = 0$, respectively. The signature of a vector is defined by $\text{sign}(x) = 1, 0$ or -1 if x is spacelike, lightlike or timelike, respectively. The *norm* of a vector $\mathbf{x} \in R_1^3$ is defined by $\|\mathbf{x}\| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$.

$\mathbf{x}$ is a *unit* spacelike (timelike) vector if $\langle \mathbf{x}, \mathbf{x} \rangle = 1$ ($\langle \mathbf{x}, \mathbf{x} \rangle = -1$) [16].

Vectors $\mathbf{x}$ and $\mathbf{y}$ are called Lorentzian orthogonal if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. Two timelike vectors cannot be orthogonal. A vector perpendicular to a timelike vector is spacelike. Timelike vectors cannot be orthogonal to null vectors. A vector perpendicular to a spacelike vector can be spacelike or timelike or null. Further, two vectors that are symmetric in the Euclidean sense with respect to the light cone are perpendicular in the Lorentzian sense.

The *cross product* of any two $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ vectors of the space R_1^3 is defined by

$$\mathbf{x} \times \mathbf{y} = \det \begin{pmatrix} -e_1 & -e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}, \quad (2.2)$$

where $\{e_1, e_2, e_3\}$ is the canonic basis of R_1^3 . For any vector $\mathbf{u}$, the equality

$$\langle \mathbf{u}, \mathbf{x} \times \mathbf{y} \rangle = \det(\mathbf{u}, \mathbf{x}, \mathbf{y}),$$

so that $\mathbf{x} \times \mathbf{y}$ is Lorentzian orthogonal to both $\mathbf{x}$ and $\mathbf{y}$. Moreover, if $\mathbf{x}$ is a unit timelike vector, $\mathbf{y}$ is a unit spacelike vector and $\langle \mathbf{x}, \mathbf{y} \rangle = 0$, $\mathbf{x} \times \mathbf{y} = \mathbf{z}$, then by a direct calculation, we have

$$\mathbf{z} \times \mathbf{x} = -\mathbf{y}, \quad \mathbf{y} \times \mathbf{z} = -\mathbf{x}, \quad \mathbf{x} \times \mathbf{y} = \mathbf{z}$$

[15, 17, 18, 19].

Lemma 2. 1 Let $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$ and $\mathbf{z} = (z_1, z_2, z_3)$ be three vectors at the space R_1^3 . In this case, the following equations are valid:

$$\text{i) } \mathbf{x} \times \mathbf{y} = -\mathbf{y} \times \mathbf{x}$$

$$\text{ii) } \langle \mathbf{x} \times \mathbf{y}, \mathbf{z} \rangle = -\det(\mathbf{x}, \mathbf{y}, \mathbf{z})$$

$$\text{iii) } (\mathbf{x} \times \mathbf{y}) \times \mathbf{z} = -\langle \mathbf{x}, \mathbf{z} \rangle \mathbf{y} + \mathbf{y} \langle \mathbf{z}, \mathbf{x} \rangle$$

$$\text{iv) } \langle \mathbf{x} \times \mathbf{y}, \mathbf{x} \rangle = 0 \text{ ve } \langle \mathbf{x} \times \mathbf{y}, \mathbf{y} \rangle = 0$$

$$\text{v) } \langle \mathbf{x} \times \mathbf{y}, \mathbf{x} \times \mathbf{y} \rangle = -\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle + (\langle \mathbf{x}, \mathbf{y} \rangle)^2 \text{ (Lagrangian identity)}$$

[15, 17, 18].

Definition 2. 2 Let $\alpha: I \rightarrow R_1^3$ be a regular curve. We say that any regular curve α is *spacelike* if for all $t \in I$, $\langle \alpha'(t), \alpha'(t) \rangle > 0$; *timelike* if $\langle \alpha'(t), \alpha'(t) \rangle < 0$, *lightlike (null)* if $\langle \alpha'(t), \alpha'(t) \rangle = 0$ [17, 19].

Definition 2. 3 Let M be a regular surface in the space R_1^3 and let $P \in M$. We say that M is *timelike surface* (resp., *spacelike surface*, *lightlike surface*) if for tangent plane $T_P M$ of the surface at point M , the induced metric

$$\langle \cdot, \cdot \rangle_P : T_P M \times T_P M \rightarrow R$$

is indefined (resp., positive definite, degenerate).

The set of all spacelike vectors of unit length centered at the origin in space R_1^3 is called the *Lorentz sphere (de Sitter 2 - space)* of radius r is denoted by

$$S_1^2(r) = \{ \mathbf{x} = (x_1, x_2, x_3) \in R_1^3 : \langle \mathbf{x}, \mathbf{x} \rangle = r^2 \}$$

If the Lorentzian scalar product is restricted on the Lorentzian sphere $S_l^2(r)$, the induced metric is undefined. This means that $S_l^2(r)$ is a timelike surface. Therefore, curves on the Lorentz sphere can have three characters.

The set of all timelike vectors of unit length centered at the origin in space R_l^3 is called the *unit hyperbolic 2 - sphere* of radius r is denoted by

$$H_o^2(r) = \{ \mathbf{x} = (x_1, x_2, x_3) \in R_l^3 : \langle \mathbf{x}, \mathbf{x} \rangle = -r^2 \}$$

The sphere $H_o^2(r)$ is the hyperboloid of two sheets of the Euclidean space E^3 . The positive (hyperbolic 2 - space) and negative sheets of this sphere are denoted by

$$H^+(r) = \{ \mathbf{x} = (x_1, x_2, x_3) \in H_o^2(r) : \langle \mathbf{x}, \mathbf{e}_3 \rangle < 0 \}$$

and

$$H^-(r) = \{ \mathbf{x} = (x_1, x_2, x_3) \in H_o^2(r) : \langle \mathbf{x}, \mathbf{e}_3 \rangle > 0 \},$$

respectively [10, 15, 17, 18, 19].

If the Lorentzian scalar product is restricted to the hyperbolic sphere $H^+(r)$, the induced metric is positive definite. This means that $H^+(r)$ is a spacelike surface.

The set of all null vectors in space R_l^3 is called the *light cone* and denoted by

$$C^2 = \{ \mathbf{x} = (x_1, x_2, x_3) \in R_l^3 : \langle \mathbf{x}, \mathbf{x} \rangle = 0, \mathbf{x} \neq (0, 0, 0) \}.$$

The positive and negative sheets of the light cone C^2 are denoted by

$$C^+ = \{ \mathbf{x} = (x_1, x_2, x_3) \in C^2 : \langle \mathbf{x}, \mathbf{e}_3 \rangle < 0 \}$$

and

$$C^{\cdot} = \{x = (x_1, x_2, x_3) \in C^2 : \langle x, e_3 \rangle > 0\},$$

respectively [10, 15, 17, 18, 19].

Definition 2. 4 i) Circular cylinder of index 1, $R_1' \times S^1$. The Cartesian equation of the of this cylinder is defined by $x_1^2 + x_2^2 = r^2$, $r > 0$. A parametrization for the cylinder is $X(u, v) = (\pm \sqrt{r^2 - v^2}, v, u)$; $0 \leq u \leq 2\pi$, $v \in R$. This is a timelike surface [17,19,20].

ii) Lorentzian circular cylinder $S_1^1 \times R$. The Cartesian equation of the surface is $x_1^2 - x_3^2 = r^2$, $r > 0$. A parametrization of this cylinder is given by

$$X(u, v) = (\pm \sqrt{r^2 + u^2}, v, u); u, v \in R.$$

This is a timelike surface [17,19,20].

iii) Hyperbolic cylinder $H^1 \times R$. The Cartesian equation of the cylinder is given by $x_1^2 - x_3^2 = -r^2$, $r > 0$. A parametrization for this cylinder is

$$X(u, v) = (\pm \sqrt{u^2 - r^2}, v, u); u, v \in R.$$

This is a spacelike surface [17,19,20].

Theorem 2. 5 Let $\alpha: I \subseteq R \rightarrow R_1^3$ be a regular timelike curve with arbitrary parameter t . In this case, the Frenet equations of the Frenet frame $\{T, N, B\}$ are given by

$$\begin{bmatrix} T'(t) \\ N'(t) \\ B'(t) \end{bmatrix} = \begin{bmatrix} 0 & v(t)\kappa(t) & 0 \\ v(t)\kappa(t) & 0 & -v(t)\tau(t) \\ 0 & v(t)\tau(t) & 0 \end{bmatrix} \begin{bmatrix} T(t) \\ N(t) \\ B(t) \end{bmatrix}, \quad (2.3)$$

where $\langle T, T \rangle = -1$, $v(t) = \|\alpha'(t)\|$,

$$T(t) = \frac{\alpha'(t)}{\|\alpha'(t)\|}, \quad B(t) = -\frac{\alpha'(t) \times \alpha''(t)}{\|\alpha'(t) \times \alpha''(t)\|}, \quad N(t) = T(t) \times B(t) \quad (2.4)$$

and

$$\kappa(t) = \frac{\left\| \frac{d\alpha(t)}{dt} \times \frac{d^2\alpha(t)}{dt^2} \right\|}{\left\| \left(\frac{d\alpha(t)}{dt} \right) \right\|^3}, \quad \tau(t) = -\frac{\det\left(\frac{d\alpha(t)}{dt}, \frac{d^2\alpha(t)}{dt^2}, \frac{d^3\alpha(t)}{dt^3}\right)}{\left\| \frac{d\alpha(t)}{dt} \times \frac{d^2\alpha(t)}{dt^2} \right\|^2}. \quad (2.5)$$

By (2.3), the instantaneous rotation vector of the Frenet frame is defined by

$$f(t) = v(t)\tau(t)T(t) - v(t)\kappa(t)B(t) \quad (2.6)$$

[15, 18, 19].

Teorem 2. 6 Let $\alpha: I \subseteq \mathbb{R} \rightarrow \mathbb{R}_l^3$ be a regular spacelike curve with arbitrary parameter t . In this case, the Frenet equations of the Frenet frame $\{T, N, B\}$ are given by

$$\begin{bmatrix} T'(t) \\ N'(t) \\ B'(t) \end{bmatrix} = \begin{bmatrix} 0 & v(t)\kappa(t) & 0 \\ v(t)\kappa(t) & 0 & v(t)\tau(t) \\ 0 & v(t)\tau(t) & 0 \end{bmatrix} \begin{bmatrix} T(t) \\ N(t) \\ B(t) \end{bmatrix}, \quad (2.7)$$

where, $\langle N, N \rangle = -1$, $v(t) = \|\alpha'(t)\|$,

$$T(t) = \frac{\alpha'(t)}{\|\alpha'(t)\|}, \quad B(t) = -\frac{\alpha'(t) \times \alpha''(t)}{\|\alpha'(t) \times \alpha''(t)\|}, \quad N(t) = B(t) \times T(t) \quad (2.8)$$

and

$$\kappa(t) = \frac{\left\| \frac{d\alpha(t)}{dt} \times \frac{d^2\alpha(t)}{dt^2} \right\|}{\left\| \frac{d\alpha(t)}{dt} \right\|^3}, \quad \tau(t) = - \frac{\det \left(\frac{d\alpha(t)}{dt}, \frac{d^2\alpha(t)}{dt^2}, \frac{d^3\alpha(t)}{dt^3} \right)}{\left\| \frac{d\alpha(t)}{dt} \times \frac{d^2\alpha(t)}{dt^2} \right\|^2}. \quad (2.9)$$

By (2.7), the instantaneous rotation vector of the Frenet frame is

$$f(t) = -v(t)\tau(t)T(t) + v(t)\kappa(t)B(t) \quad (2.10)$$

[15,18,19].

Theorem 2. 7 Let $\alpha: I \subseteq \mathbb{R} \rightarrow R_1^3$ be a regular spacelike curve with arbitrary parameter t . In this case, the Frenet equations of the Frenet trihedron $\{T, N, B\}$ are given by

$$\begin{bmatrix} T'(t) \\ N'(t) \\ B'(t) \end{bmatrix} = \begin{bmatrix} 0 & v(t)\kappa(t) & 0 \\ -v(t)\kappa(t) & 0 & v(t)\tau(t) \\ 0 & v(t)\tau(t) & 0 \end{bmatrix} \begin{bmatrix} T(t) \\ N(t) \\ B(t) \end{bmatrix}, \quad (2.11)$$

where $\langle B, B \rangle = -1$, $v(t) = \|\alpha'(t)\|$,

$$T(t) = \frac{\alpha'(t)}{\|\alpha'(t)\|}, \quad B(t) = \frac{\alpha'(t) \times \alpha''(t)}{\|\alpha'(t) \times \alpha''(t)\|}, \quad N(t) = T(t) \times B(t) \quad (2.12)$$

and the curvature functions are given by (2.9).

By (2.11), the instantaneous rotation vector of the Frenet trihedron is

$$f(t) = v(t)\tau(t)T(t) - v(t)\kappa(t)B(t) \quad (2.13)$$

[15,18,19].

3 Lorentzian Viviani Curve I

In this chapter, we define the parametric equations and examine the geometry of the curve which is the intersection of the Lorentzian sphere of radius a centered at the origin with the right circular cylinder of radius a centered at $(2a, 0, 0)$ in the Lorentz-Minkowski 3 - space.

A parameterization for this curve is given by

$$\alpha_1(t) = \left(a(2 + \cos t), a \sin t, \pm 2\sqrt{2} a \cos \frac{t}{2} \right), \quad t \in [-2\pi, 2\pi], \quad (3.1)$$

where a is positive. The curve defined by (3.1) is called *Lorentzian Viviani curve I*.

For this curve, we have

$$v_1(t) = \|\alpha_1'(t)\| = \sqrt{|\langle \alpha_1'(t), \alpha_1'(t) \rangle|} = a\sqrt{|\cos t|} \quad (3.2)$$

from

$$\alpha_1'(t) = \left(-a \sin t, a \cos t, \mp \sqrt{2} a \sin \frac{t}{2} \right) \quad (3.3)$$

and

$$\langle \alpha_1'(t), \alpha_1'(t) \rangle = a^2 \left(1 - 2 \left(\frac{1 - \cos t}{2} \right) \right) = a^2 \cos t.$$

That is, $\alpha_1(t)$ is not parametrized by the arc length parameter. If $\cos t > 0$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$, the curve arc is spacelike, if $\cos t < 0$, $\frac{\pi}{2} < t < \frac{3\pi}{2}$, the curve arc is timelike, and if $\cos t = 0$, $t = \frac{(2k+1)\pi}{2}$, $k \in \mathbb{Z}$, the tangents of the curve are lightlike vectors.

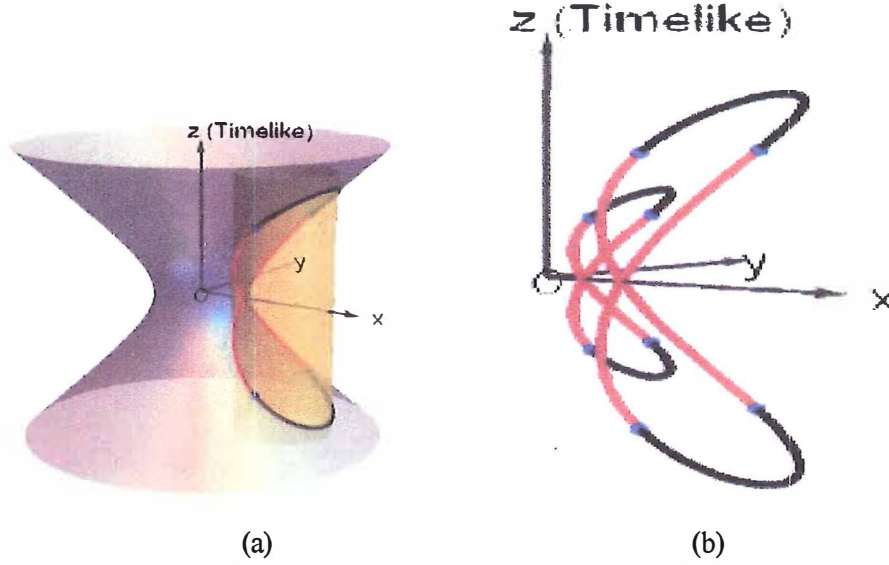


Fig. 3. 1: Lorentzian Viviani Curve I, spacelike (black) and timelike (red) arcs, and lightlike points (blue), (a) $a=1$, (b) $a=1$ and $a=2$.

Theorem 3. 1 Let $\alpha_t : I \subseteq \mathbb{R} \rightarrow S_t^2(a)$ be regular Lorentzian Viviani curve I with arbitrary parameter t . In this case, the Frenet equations of the Frenet frame $\{T, N, B\}$ are given by

$$\begin{bmatrix} T'(t) \\ N'(t) \\ B'(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{\sqrt{|1-3\cos t|}}{2a\cos t} & 0 \\ \frac{\varepsilon\sqrt{|1-3\cos t|}}{2a\cos t} & 0 & \mp \frac{3\sqrt{2}\sin\frac{t}{2}\sqrt{\cos t}}{a|1-3\cos t|} \\ 0 & \mp \frac{3\sqrt{2}\sin\frac{t}{2}\sqrt{\cos t}}{a|1-3\cos t|} & 0 \end{bmatrix} \begin{bmatrix} T(t) \\ N(t) \\ B(t) \end{bmatrix} \quad (3.4)$$

for $-\frac{\pi}{2} < t < \frac{\pi}{2}$ and

$$\begin{bmatrix} T'(t) \\ N'(t) \\ B'(t) \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\sqrt{1-3\cos t}}{2a\cos t} & 0 \\ \frac{\sqrt{1-3\cos t}}{2a\cos t} & 0 & \pm \frac{3\sqrt{2}\sin\frac{t}{2}\sqrt{-\cos t}}{a(1-3\cos t)} \\ 0 & \mp \frac{3\sqrt{2}\sin\frac{t}{2}\sqrt{-\cos t}}{a(1-3\cos t)} & 0 \end{bmatrix} \begin{bmatrix} T(t) \\ N(t) \\ B(t) \end{bmatrix} \quad (3.5)$$

for $\frac{\pi}{2} < t < \frac{3\pi}{2}$.

Proof. By (3.2) and (3.3) the unit tangent vector field is

$$T(t) = \frac{1}{\sqrt{|\cos t|}} \left(-\sin t, \cos t, \mp \sqrt{2} \sin \frac{t}{2} \right).$$

i) The curve $\alpha_1(t)$ is spacelike in the interval $-\frac{\pi}{2} < t < \frac{\pi}{2}$. Thus, we have

$$T(t) = \frac{1}{\sqrt{\cos t}} \left(-\sin t, \cos t, \mp \sqrt{2} \sin \frac{t}{2} \right), \quad (3.6)$$

$$T'(t) = \left(-\frac{\cos^2 t + 1}{2(\cos t)^{3/2}}, -\frac{\sin t}{2\sqrt{\cos t}}, \mp \frac{\sqrt{2}}{2} \frac{\cos \frac{t}{2}}{(\cos t)^{3/2}} \right) \quad (3.7)$$

and

$$\begin{aligned}\langle T'(t), T'(t) \rangle &= \frac{1}{4} (\cos^2 t + 1)^2 \sec^3(t) + \frac{1}{4} \sin^2 t \sec t - \frac{1}{2} \cos^2 \frac{t}{2} \sec^3 t \\ &= \frac{1}{4} \sec t (3 - \sec t).\end{aligned}$$

The solution of the equation

$$\langle T'(t), T'(t) \rangle = 0 \Rightarrow \frac{1}{4} \sec t (3 - \sec t) = 0$$

is $t = \pm \arccos \frac{1}{3}$. In these points, the vectors $T'(t)$ are lightlike (Figure 3. 2).

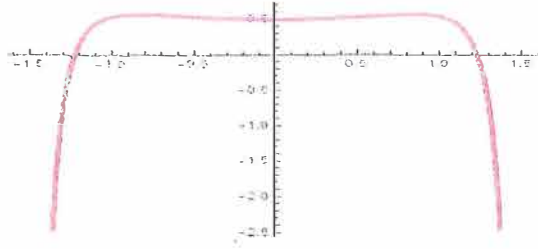


Fig. 3. 2: The function $\langle T'(t), T'(t) \rangle$ in $-\frac{\pi}{2} < t < \frac{\pi}{2}$

By a direct calculation, we have

$$\alpha_1''(t) = \left(-a \cos t, -a \sin t, \mp \frac{\sqrt{2}}{2} a \cos \frac{t}{2} \right),$$

$$\alpha_1'''(t) = \left(a \sin t, -a \cos t, \pm \frac{\sqrt{2}}{4} a \sin \frac{t}{2} \right).$$

If we find the cross product of the first and second derivatives and take the length of the vector, we obtain

$$\|\alpha_1'(t) \times \alpha_1''(t)\| = \sqrt{|\langle \alpha_1'(t) \times \alpha_1''(t), \alpha_1'(t) \times \alpha_1''(t) \rangle|} = \frac{a^2}{\sqrt{2}} \sqrt{\left| 3 \sin^2 \frac{t}{2} - 1 \right|}$$

from

$$\begin{aligned} \alpha_1'(t) \times \alpha_1''(t) &= \begin{vmatrix} -e_1 & -e_2 & e_3 \\ -a \sin t & a \cos t & \mp \sqrt{2} a \sin \frac{t}{2} \\ -a \cos t & -a \sin t & \mp \frac{\sqrt{2}}{2} a \cos \frac{t}{2} \end{vmatrix}, \\ &= \left(\pm \frac{\sqrt{2}}{2} a^2 \cos t \cos \frac{t}{2} \pm \sqrt{2} a^2 \sin t \sin \frac{t}{2}, \pm \frac{\sqrt{2}}{2} a^2 \sin t \cos \frac{t}{2} \mp \sqrt{2} a^2 \cos t \sin \frac{t}{2}, a^2 \right), \\ \langle \alpha_1'(t) \times \alpha_1''(t), \alpha_1'(t) \times \alpha_1''(t) \rangle &= \left(\pm \frac{\sqrt{2}}{2} a^2 \cos t \cos \frac{t}{2} \pm \sqrt{2} a^2 \sin t \sin \frac{t}{2} \right)^2 + \\ &\quad \left(\pm \frac{\sqrt{2}}{2} a^2 \sin t \cos \frac{t}{2} \mp \sqrt{2} a^2 \cos t \sin \frac{t}{2} \right)^2 - a^4 \\ &= \frac{a^4}{2} \left(3 \sin^2 \frac{t}{2} - 1 \right). \end{aligned}$$

Then, it is seen that $t = \pm \arccos \frac{1}{3}$ from

$$\langle \alpha_1'(t) \times \alpha_1''(t), \alpha_1'(t) \times \alpha_1''(t) \rangle = 0.$$

(Figure 3. 3)

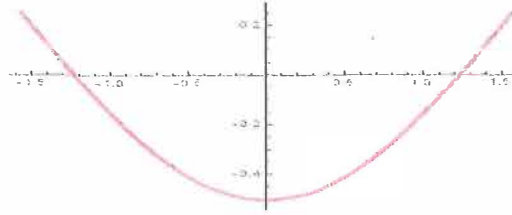


Fig. 3. 3: The function $\langle \alpha_1'(t) \times \alpha_1''(t), \alpha_1'(t) \times \alpha_1''(t) \rangle$, $a=1$.

Therefore, the unit binormal vector field is

$$B(t) = \frac{1}{\sqrt{|1-3\cos t|}} \left(\pm \frac{\cos \frac{3t}{2} - 3\cos \frac{t}{2}}{\sqrt{2}}, \mp 2\sqrt{2} \sin^3 \frac{t}{2}, -2 \right). \quad (3.8)$$

Lets denote compenents of unit pricipale normal vector field

$$N(t) = \frac{\sqrt{2}}{\sqrt{\cos t} \sqrt{|3\sin^2 \frac{t}{2} - 1|}} \begin{vmatrix} -e_1 & -e_2 & e_3 \\ \mp \frac{\sqrt{2}}{2} \cos t \cos \frac{t}{2} \mp \sqrt{2} \sin t \sin \frac{t}{2} & \mp \frac{\sqrt{2}}{2} \sin t \cos \frac{t}{2} \pm \sqrt{2} \cos t \sin \frac{t}{2} & -1 \\ -\sin t & \cos t & \mp \sqrt{2} \sin \frac{t}{2} \end{vmatrix}$$

by N_i , $i=1, 2, 3$. Thus, we find

$$\begin{aligned} N_1 &= \frac{\sqrt{2}}{\sqrt{\cos t} \sqrt{|3\sin^2 \frac{t}{2} - 1|}} \left(-\sin t \sin \frac{t}{2} \cos \frac{t}{2} + 2 \cos t \sin \frac{t}{2} \sin \frac{t}{2} - \cos t \right) \\ &= -\frac{1 + \cos^2 t}{\sqrt{2} \sqrt{\cos t} \sqrt{|3\sin^2 \frac{t}{2} - 1|}}, \end{aligned}$$

$$N_2 = \frac{\sqrt{2}}{\sqrt{\cos t} \sqrt{\left| 3 \sin^2 \frac{t}{2} - 1 \right|}} \left(\cos t \cos \frac{t}{2} \sin \frac{t}{2} + 2 \sin t \sin \frac{t}{2} \sin \frac{t}{2} - \sin t \right)$$

$$= - \frac{\sin t \cos t}{\sqrt{2} \sqrt{\cos t} \sqrt{\left| 3 \sin^2 \frac{t}{2} - 1 \right|}}$$

and

$$N_3 = \frac{\sqrt{2}}{\sqrt{\cos t} \sqrt{\left| 3 \sin^2 \frac{t}{2} - 1 \right|}} \left(\mp \frac{\sqrt{2}}{2} \cos t \cos t \cos \frac{t}{2} \mp \sqrt{2} \sin t \cos t \sin \frac{t}{2} \mp \frac{\sqrt{2}}{2} \sin t \sin t \cos \frac{t}{2} \pm \sqrt{2} \sin t \cos t \sin \frac{t}{2} \right)$$

$$= \mp \frac{\cos \frac{t}{2}}{\sqrt{\cos t} \sqrt{\left| 3 \sin^2 \frac{t}{2} - 1 \right|}}.$$

Consequently, we have

$$N(t) = \frac{1}{\sqrt{\cos t} \sqrt{\left| 3 \sin^2 \frac{t}{2} - 1 \right|}} \left(-\frac{1 + \cos^2 t}{\sqrt{2}}, -\frac{\sin t \cos t}{\sqrt{2}}, \mp \cos \frac{t}{2} \right)$$

or

$$N(t) = \frac{\sqrt{\cos t}}{\sqrt{|1 - 3 \cos t|}} \left(-\frac{(\cos 2t + 3) \sec t}{2}, -\sin t, \mp \sqrt{2} \cos \frac{t}{2} \sec t \right). \quad (3.9)$$

By (2.9), the curvature and torsion functions are found in the forms

$$\kappa(t) = \frac{\frac{a^2}{\sqrt{2}} \sqrt{\left| 3 \sin^2 \frac{t}{2} - 1 \right|}}{a^3 (\cos t)^{3/2}}$$

$$\kappa(t) = \frac{\sqrt{|1 - 3 \cos t|}}{2a (\cos t)^{3/2}} \quad (3.10)$$

and

$$\tau(t) = \mp \frac{3\sqrt{2}}{4} a^3 \sin \frac{t}{2} \frac{2}{a^4 \left| 3 \sin^2 \frac{t}{2} - 1 \right|}$$

$$\tau(t) = \mp \frac{3\sqrt{2} \sin \frac{t}{2}}{a |1 - 3 \cos t|} \quad (3.11)$$

from

$$\begin{aligned} \langle \alpha_1'(t) \times \alpha_1''(t), \alpha_1''' \rangle &= \pm \frac{\sqrt{2}}{2} a^3 \sin t \cos t \cos \frac{t}{2} \pm \sqrt{2} a^3 \sin^2 t \sin \frac{t}{2} \mp \\ &\quad \frac{\sqrt{2}}{2} a^3 \sin t \cos t \cos \frac{t}{2} \pm \sqrt{2} a^3 \cos^2 t \sin \frac{t}{2} \mp \frac{\sqrt{2}}{4} a^3 \sin \frac{t}{2} \\ &= \pm \sqrt{2} a^3 \sin \frac{t}{2} \mp \frac{\sqrt{2}}{4} a^3 \sin \frac{t}{2} \\ &= \pm \frac{3\sqrt{2}}{4} a^3 \sin \frac{t}{2}, \end{aligned}$$

respectively.

For $t = 0$ the torsion is zero. Since the torsion function measures the amount of deviation of the curve from the tangent plane, the sign here indicates that the curve is on the opposite side of the tangent plane (Figure 3. 4).

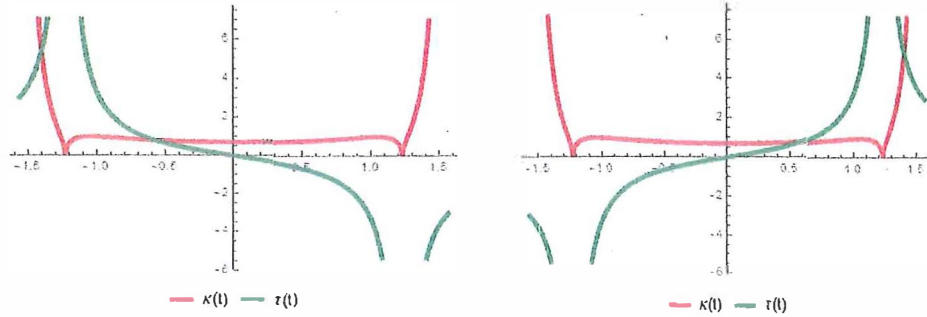


Fig. 3. 4: Curvature and torsion functions of the curve $\alpha_1(t)$ at the open interval $-\frac{\pi}{2} < t < \frac{\pi}{2}$, $\alpha=1$.

From figure 3. 2 and figure 3. 3, the vector $T'(t)$ is spacelike in interval $\left(-\arccos \frac{1}{3}, \arccos \frac{1}{3}\right)$. In this domain, the characters of the principle normal and binormal change while the curvature and torsion functions do not change. Now, let's take the spacelike arc of the curve. Then, the Frenet derivative formulas of the frame $\{T(t), N(t), B(t)\}$ are given by

$$\begin{bmatrix} T'(t) \\ N'(t) \\ B'(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{\sqrt{|1-3\cos t|}}{2\cos t} & 0 \\ \frac{\varepsilon\sqrt{|1-3\cos t|}}{2\cos t} & 0 & \mp \frac{3\sqrt{2}\sin \frac{t}{2}\sqrt{\cos t}}{|1-3\cos t|} \\ 0 & \mp \frac{3\sqrt{2}\sin \frac{t}{2}\sqrt{\cos t}}{|1-3\cos t|} & 0 \end{bmatrix} \begin{bmatrix} T(t) \\ N(t) \\ B(t) \end{bmatrix} \quad (3.12)$$

from equations (3.10) and (3.11), with $\varepsilon = 1$ at the interval where the vector $T'(t)$ is timelike and $\varepsilon = -1$ at the interval where it is spacelike.

Corollary 3. 2 For $t \in \left(-\frac{\pi}{2}, -\arccos\frac{1}{3}\right) \cup \left(\arccos\frac{1}{3}, \frac{\pi}{2}\right)$, the vector $T'(t)$ is timelike. Hence, the instantaneous rotation vector of the Frenet frame of the spacelike arc of the curve $\alpha_1(t)$ is given by

$$f(t) = \pm 3\sqrt{2} \frac{\sqrt{\cos t} \sin \frac{t}{2}}{1 - 3\cos t} T(t) + \frac{\sqrt{1 - 3\cos t}}{2\cos t} B(t). \quad (3.13)$$

For $t \in \left(-\arccos\frac{1}{3}, \arccos\frac{1}{3}\right)$, the vector $T'(t)$ is spacelike. Then the instantaneous rotation vectors of the Frenet frame of the spacelike arc of the curve $\alpha_1(t)$ is

$$f_*(t) = \mp 3\sqrt{2} \frac{\sqrt{\cos t} \sin \frac{t}{2}}{3\cos t - 1} T(t) - \frac{\sqrt{3\cos t - 1}}{2\cos t} B(t). \quad (3.14)$$

When the + sign is taken in (3.1), the Frenet trihedron and instantaneous rotation vectors are seen in figure 3. 5.

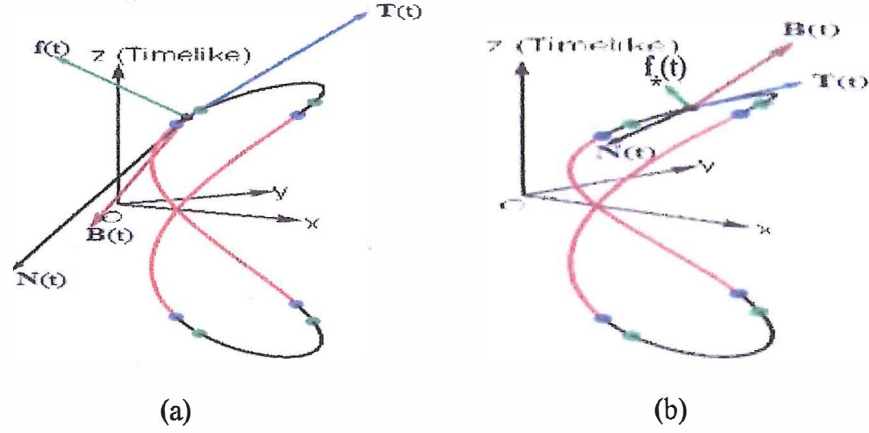


Fig. 3. 5: (a) The Frenet trihedron and instantaneous rotation vectors of the spacelike arc of the curve $\alpha_1(t)$, $T'(t)$ is timelike, $t \in \left(-\frac{\pi}{2}, -\cos^{-1}\frac{1}{3}\right)$,

(b) The Frenet trihedron and instantaneous rotation vectors of the spacelike arc of the curve $\alpha_1(t)$, $T'(t)$ is spacelike $t \in \left(-\cos^{-1}\frac{1}{3}, \cos^{-1}\frac{1}{3}\right)$.

Since the curve $\alpha_1(t)$ is not of single character, both non-null frames and null frames are defined on it. Null frames are defined for $t = \cos^{-1}\frac{1}{3}$, green colored points and $t = \frac{(2k+1)\pi}{2}$, $k \in \mathbb{Z}$, blue colored points as seen in figure 3.5. The frame at these points are similar to the null frames in [24, 25]. However at the blue colored points on the curve $\alpha_1(t)$, the character of the curve changes and at the green colored points, the characteristics of the vectors N and B exchange.

ii) Let $\frac{\pi}{2} < t < \frac{3\pi}{2}$. In this case, we have

$$T(t) = \frac{1}{\sqrt{-\cos t}} \left(-\sin t, \cos t, \mp \sqrt{2} \sin \frac{t}{2} \right) \quad (3.15)$$

and

$$\langle T(t), T(t) \rangle = \frac{1 - 2 \sin^2 \frac{t}{2}}{-\cos t} = \frac{\cos t}{-\cos t} = -1 < 0.$$

This means that the curve $\alpha_t(t)$ is timelike.

For the vector $T(t)$ we find

$$\begin{aligned} T'(t) &= \left(-\frac{\cos t}{\sqrt{-\cos t}} + \frac{\sin^2 t}{2(-\cos t)^{3/2}}, -\frac{\sin t}{\sqrt{-\cos t}} \frac{\cos t \sin t}{2(-\cos t)^{3/2}}, -\frac{\cos \frac{t}{2}}{\sqrt{2}\sqrt{-\cos t}} + \frac{\sin \frac{t}{2} \sin t}{\sqrt{2}(-\cos t)^{3/2}} \right) \\ T'(t) &= \frac{1}{\sqrt{2} \cos t \sqrt{-\cos t}} \left(\frac{1 + \cos^2 t}{\sqrt{2}}, \frac{\sin t \cos t}{\sqrt{2}}, \pm \cos \frac{t}{2} \right) \\ T'(t) &= \left(\frac{1}{2} \sqrt{-\cos t} (2 + \tan^2 t), \frac{\sin t}{2\sqrt{-\cos t}}, \mp \frac{\cos \frac{t}{2} + \sin \frac{t}{2} \tan t}{\sqrt{2}\sqrt{-\cos t}} \right) \end{aligned} \quad (3.16)$$

and

$$\begin{aligned} \langle T'(t), T'(t) \rangle &= -\frac{1}{4} \sin t \tan t - \frac{1}{4} \cos t (2 + \tan^2 t)^2 + \frac{1}{2} \sec t \left(\cos \frac{t}{2} + \sin \frac{t}{2} \tan t \right)^2 \\ &= \frac{1}{4} (-3 + \sec t) \sec t. \end{aligned}$$

Thus, the vector $T'(t)$ is spacelike in the interval $\frac{\pi}{2} < t < \frac{3\pi}{2}$ (Figure 3. 6).

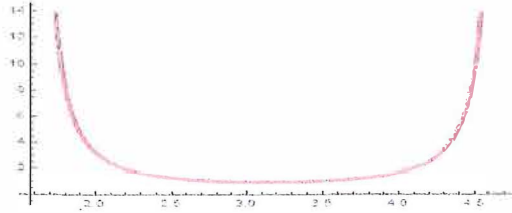


Fig. 3. 6: The function $\langle T'(t), T'(t) \rangle$ in $\frac{\pi}{2} < t < \frac{3\pi}{2}$.

Since $3\sin^2 \frac{t}{2} - 1 = \frac{1-3\cos t}{2} > 0$ in interval $\frac{\pi}{2} < t < \frac{3\pi}{2}$, the vector field

$$B(t) = \frac{1}{\sqrt{1-3\cos t}} \left(\pm \frac{\cos \frac{3t}{2} - 3\cos \frac{t}{2}}{\sqrt{2}}, \mp 2\sqrt{2}\sin^3 \frac{t}{2}, -2 \right) \quad (3.17)$$

is spacelike and the principle normal vector field is given by

$$N(t) = \frac{\sqrt{-\cos t}}{\sqrt{1-3\cos t}} \left(\frac{(\cos 2t + 3)\sec t}{2}, \sin t, \pm \sqrt{2} \cos \frac{t}{2} \sec t \right). \quad (3.18)$$

By (2.9), the curvature and torsion functions of curve are obtained in the forms given by

$$\kappa(t) = \frac{\sqrt{1-3\cos t}}{2a(-\cos t)^{3/2}} \quad (3.19)$$

and

$$\tau(t) = \pm \frac{3\sqrt{2} \sin \frac{t}{2}}{a(1-3\cos t)}, \quad (3.20)$$

respectively (Figure 3. 7).

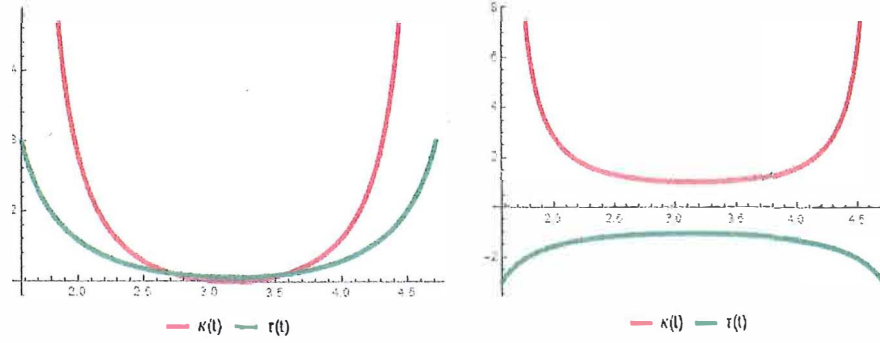


Fig. 3. 7: The curvature and torsion functions of the timelike arc of the curve $\alpha_t(t)$, $\frac{\pi}{2} < t < \frac{3\pi}{2}$, $a=1$.

Thus, the Frenet formulas of the timelike arc of the curve $\alpha_t(t)$ is given by

$$\begin{bmatrix} T'(t) \\ N'(t) \\ B'(t) \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\sqrt{1-3\cos t}}{2\cos t} & 0 \\ \frac{\sqrt{1-3\cos t}}{2\cos t} & 0 & \pm \frac{3\sqrt{2} \sin \frac{t}{2} \sqrt{-\cos t}}{1-3\cos t} \\ 0 & \mp \frac{3\sqrt{2} \sin \frac{t}{2} \sqrt{-\cos t}}{1-3\cos t} & 0 \end{bmatrix} \begin{bmatrix} T(t) \\ N(t) \\ B(t) \end{bmatrix}. \quad (3.21)$$

Corollary 3. 3 The Frenet instantaneous rotation vector of the timelike arc of the curve $\alpha_t(t)$ in the interval $\frac{\pi}{2} < t < \frac{3\pi}{2}$ can be written in the form

$$f_1(t) = \pm \frac{3\sqrt{2}\sqrt{-\cos t} \sin \frac{t}{2}}{1-3\cos t} T(t) - \frac{\sqrt{-\cos t} \sqrt{1-3\cos t}}{2(-\cos t)^{3/2}} B(t) \quad (3.22)$$

(Figure 3. 8).

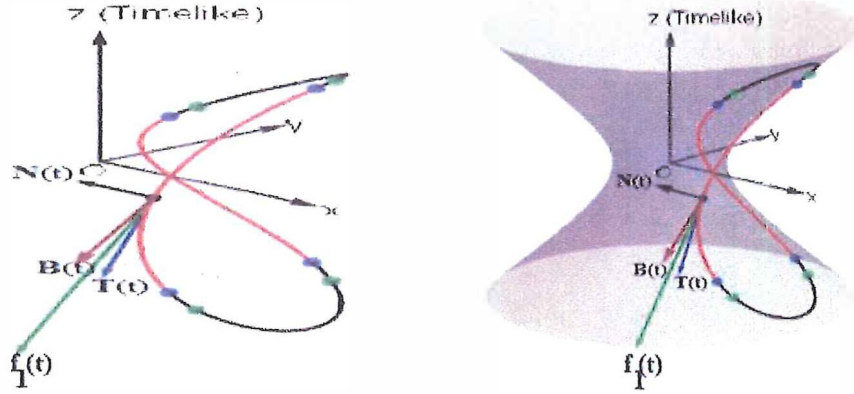


Fig. 3. 8: Frenet trihedron and the instantaneous rotation vector of the timelike arc of the curve $\alpha_1(t)$

Corollary 3. 4 The Frenet formulas of the timelike arc of the curve $\alpha_1(t)$ can be written in the form

$$T'(t) = f_1(t) \times T(t), \quad N'(t) = f_1(t) \times N(t), \quad B'(t) = f_1(t) \times B(t)$$

by the vector $f_1(t)$.

Proof. It is sufficient to multiply the Frenet vectors by $f_1(t)$ vectorically in the sense of Lorentz.

4 Lorentzian Viviani Curve II

In this chapter, we define the parametric equations and examine the geometry of the curve which is the intersection of the Lorentzian sphere of the radius a centered at the origin with Lorentzian cylinder of radius a centered at $(2a, 0, 0)$ and radius a at the Lorentz-Minkowski 3 - space.

A parametrization for this curve is given by

$$\alpha_2(t) = \left(a(2 - \cosh t), \pm 2\sqrt{2}a \sinh \frac{t}{2}, a \sinh t \right), \quad t \in \mathbb{R} \quad (4.1)$$

where a is positive. The curve defined by (4.1) is called *Lorentzian Viviani curve II*.

For the curve (4.1), we have

$$v_2(t) = \|\alpha_2'(t)\| = \sqrt{|\langle \alpha_2'(t), \alpha_2'(t) \rangle|} = a\sqrt{\cosh t} \quad (4.2)$$

from

$$\alpha_2'(t) = \left(-a \sinh t, \pm \sqrt{2}a \cosh \frac{t}{2}, a \cosh t \right). \quad (4.3)$$

By (4.2), $\alpha_2(t)$ is spacelike (Figure 4. 1) and it is not parametrized by the arc length.

If the second component of the curve $\alpha_2(t)$ is taken as $+$, the green colored curve arc is obtained and if it is taken as $-$, the red curve arc is obtained, as seen in figure (4.1).

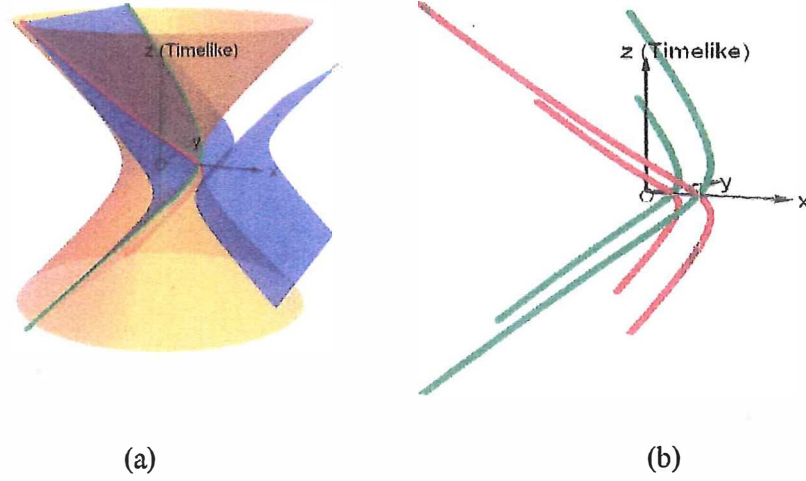


Fig. 4. 1: Lorentzian Viviani curve II (a) $a=1$, (b) $a=1$ and $a=2$.

Theorem 4. 1 Let $\alpha_2 : I \subseteq \mathbb{R} \rightarrow S^2_1(a)$ be regular spacelike Lorentzian Viviani curve II with arbitrary parameter t . In this case, the Frenet formulae of the Frenet frame $\{T, N, B\}$ are given by

$$\left. \begin{aligned} T'(t) &= \frac{1}{2} \frac{\sqrt{3 \cosh t + 1}}{\cosh^2 t} N(t) \\ N'(t) &= -\frac{1}{2} \frac{\sqrt{3 \cosh t + 1}}{\cosh^2 t} T(t) \pm 3\sqrt{2} \frac{\cosh \frac{t}{2} \sqrt{\cosh t}}{3 \cosh t + 1} B(t) \\ B'(t) &= \pm 3\sqrt{2} \frac{\cosh \frac{t}{2} \sqrt{\cosh t}}{3 \cosh t + 1} N(t) \end{aligned} \right\} \quad (4.4)$$

Proof. By (4.2) and (4.3) we have

$$T(t) = \frac{1}{\sqrt{\cosh t}} \left(-\sinh t, \pm \sqrt{2} \cosh \frac{t}{2}, \cosh t \right). \quad (4.5)$$

The derivatives of components of vector field $T(t)$ are given by

$$\begin{aligned} \frac{d}{dt} \left(-\frac{\sinh t}{\sqrt{\cosh t}} \right) &= -\frac{\cosh t \sqrt{\cosh t} - \sinh^2 t}{2 \cosh^{3/2} t} = \frac{\sinh^2 t - 2 \cosh^2 t}{2 \cosh^{3/2} t} = -\frac{\cosh^2 t + 1}{2 \cosh^{3/2} t}, \\ \frac{d}{dt} \left(\pm \frac{\sqrt{2} \cosh \frac{t}{2}}{\sqrt{\cosh t}} \right) &= \pm \frac{\sinh \frac{t}{2} \cosh t - \sinh t \cosh \frac{t}{2}}{\sqrt{2} \cosh^{3/2} t} = \mp \frac{\sinh \frac{t}{2}}{\sqrt{2} \cosh^{3/2} t} \end{aligned}$$

and

$$\frac{d}{dt} \left(\frac{\cosh t}{\sqrt{\cosh t}} \right) = \frac{2 \sinh t \cosh t - \sinh t \cosh t}{2 \cosh^{3/2} t} = \frac{\sinh t \cosh t}{2 \cosh^{3/2} t} = \frac{\sinh t}{2\sqrt{\cosh t}},$$

respectively. Therefore, we have

$$T'(t) = \left(-\frac{\cosh^2 t + 1}{2(\cosh t)^{3/2}}, \mp \frac{\sinh \frac{t}{2}}{\sqrt{2}(\cosh t)^{3/2}}, \frac{\sinh t}{2\sqrt{\cosh t}} \right) \quad (4.6)$$

and

$$\begin{aligned} \langle T'(t), T'(t) \rangle &= \left(\frac{\sinh^2 t - 2 \cosh^2 t}{2 \cosh^{3/2} t} \right)^2 + \frac{\sinh^2 \frac{t}{2}}{2 \cosh^3 t} - \frac{\sinh^2 t}{4 \cosh t} \\ &= \frac{\cosh t (3 \cosh t + 1)}{4 \cosh^3 t}. \end{aligned}$$

Since $\cosh t > 1$, $T'(t)$ is spacelike and it seen that

$$\|T'(t)\| = \sqrt{-\frac{\cosh t(3\cosh t + 1)}{4\cosh^3 t}} = \frac{1}{2} \frac{\sqrt{3\cosh t + 1}}{\cosh t}.$$

By a straightforward calculation, we have

$$\alpha_2''(t) = \left(-a \cosh t, \pm \frac{a}{\sqrt{2}} \sinh \frac{t}{2}, a \sinh t \right)$$

and

$$\begin{aligned} \alpha_2'(t) \times \alpha_2''(t) &= \begin{vmatrix} -e_1 & -e_2 & e_3 \\ -a \sinh t & \pm \sqrt{2} a \cosh \frac{t}{2} & a \cosh t \\ -a \cosh t & \pm \frac{a}{\sqrt{2}} \sinh \frac{t}{2} & a \sinh t \end{vmatrix} \\ &= a^2 \left(\mp \sqrt{2} \sinh t \cosh \frac{t}{2} \pm \frac{1}{\sqrt{2}} \cosh t \sinh \frac{t}{2}, 1, \mp \frac{1}{\sqrt{2}} \sinh t \sinh \frac{t}{2} \pm \sqrt{2} \cosh t \cosh \frac{t}{2} \right) \\ &= a^2 \left(\mp \frac{\sqrt{2}}{4} \sinh \frac{3t}{2} \mp \frac{3\sqrt{2}}{4} \sinh \frac{t}{2}, 1, \pm \frac{\sqrt{2}}{4} \cosh \frac{3t}{2} \pm \frac{3\sqrt{2}}{4} \cosh \frac{t}{2} \right) \\ \alpha_2'(t) \times \alpha_2''(t) &= a^2 \left(\mp \frac{\sqrt{2}}{4} \sinh \frac{3t}{2} \mp \frac{3\sqrt{2}}{4} \sinh \frac{t}{2}, 1, \pm \sqrt{2} \cosh^3 \frac{t}{2} \right). \end{aligned}$$

Therefore, it is seen that

$$\|\alpha_2'(t) \times \alpha_2''(t)\| = \frac{a^2}{2} \sqrt{3\cosh t + 1}.$$

Consequently, by (2.9) the curvature function and unit principle normal vector field are found in the forms

$$\kappa(t) = \frac{\|T'(t)\|}{\|a_2'(t)\|} = \frac{\frac{1}{2} \frac{\sqrt{3 \cosh t + 1}}{\cosh t}}{a \sqrt{\cosh t}} = \frac{1}{2} \frac{\sqrt{3 \cosh t + 1}}{a (\cosh t)^{3/2}}.$$

$$\kappa(t) = \frac{1}{2} \frac{\sqrt{3 \cosh t + 1}}{a (\cosh t)^{3/2}}. \quad (4.7)$$

and

$$N(t) = \frac{2 \cosh t}{\sqrt{3 \cosh t + 1}} \left(\frac{\sinh^2 t - 2 \cosh^2 t}{2 (\cosh t)^{3/2}}, \mp \frac{\sinh \frac{t}{2}}{\sqrt{2} (\cosh t)^{3/2}}, \frac{\sinh t}{2 \sqrt{\cosh t}} \right)$$

$$= \left(\frac{\sinh^2 t - 2 \cosh^2 t}{\sqrt{\cosh t} \sqrt{3 \cosh t + 1}}, \mp \frac{\sqrt{2} \sinh \frac{t}{2}}{\sqrt{\cosh t} \sqrt{3 \cosh t + 1}}, \frac{\sinh t \cosh t}{\sqrt{\cosh t} \sqrt{3 \cosh t + 1}} \right)$$

$$N(t) = \left(-\frac{\sqrt{\cosh t} (\cosh 2t + 3) \operatorname{sech} t}{2 \sqrt{3 \cosh t + 1}}, \mp \frac{\sqrt{2} \sqrt{\cosh t} \sinh \frac{t}{2} \operatorname{sech} t}{\sqrt{3 \cosh t + 1}}, \frac{\sqrt{\cosh t} \sinh t}{\sqrt{3 \cosh t + 1}} \right). \quad (4.8)$$

The cross product of the vector fields $T(t)$ and $N(t)$ give the unit binormal vector field

$$B(t) = \frac{1}{\sqrt{3 \cosh t + 1}} \left(\mp \frac{\sinh \frac{3t}{2} + 3 \sinh \frac{t}{2}}{\sqrt{2}}, 2, \pm 2 \sqrt{2} \cosh^3 \frac{t}{2} \right). \quad (4.9)$$

Since $\langle B(t), B(t) \rangle = -1$, $B(t)$ is timelike. If the third derivative of the curve $\alpha_2(t)$ is found and take the mixed product of the first three derivatives is taken, it is seen that

$$\begin{aligned} \det(\alpha_2'(t), \alpha_2''(t), \alpha_2'''(t)) &= -\langle \alpha_2'(t) \times \alpha_2''(t), \alpha_2'''(t) \rangle \\ &= -a^3 \left(\pm \sqrt{2} \sinh^2 t \cosh \frac{t}{2} \mp \frac{1}{\sqrt{2}} \sinh t \cosh t \sinh \frac{t}{2} \pm \frac{1}{2\sqrt{2}} \cosh \frac{t}{2} \right. \\ &\quad \left. \pm \frac{1}{\sqrt{2}} \sinh t \cosh t \sinh \frac{t}{2} \mp \sqrt{2} \cosh^2 t \cosh \frac{t}{2} \right) \\ &= \pm a^3 \frac{3}{2\sqrt{2}} \cosh \frac{t}{2}. \end{aligned}$$

This means that the torsion function is

$$\tau(t) = \pm \frac{3\sqrt{2} \cosh \frac{t}{2}}{a(3 \cosh t + 1)} \quad (4.10)$$

(Figure 4. 2). Consequently, the Frenet equations are valid.

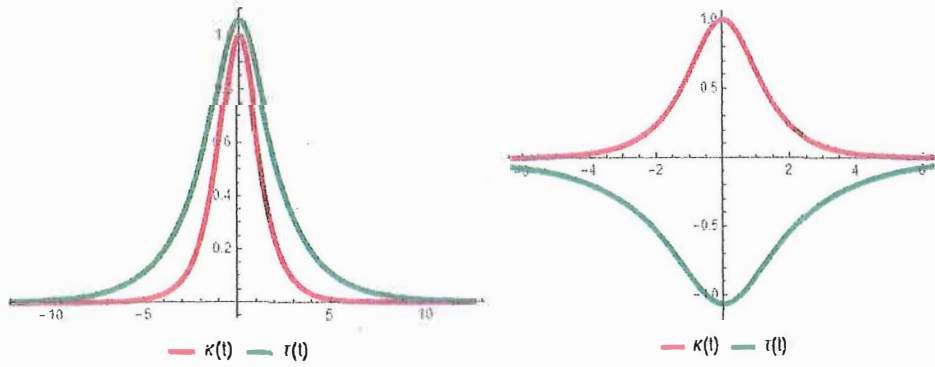


Fig. 4. 2: Curvature and torsion functions of Lorentzian Viviani curve II, $\alpha = 1$.

Corollary 4. 2 The Frenet instantaneous rotation vector of the frame $\{T(t), N(t), B(t)\}$ of the curve $\alpha_2(t)$ can be written in the form

$$f_2(t) = \pm 3\sqrt{2} \frac{\cosh \frac{t}{2} \sqrt{\cosh t}}{3 \cosh t + 1} T(t) - \frac{1}{2} \frac{\sqrt{3 \cosh t + 1}}{\cosh t} B(t). \quad (4.11)$$

Corollary 4. 3 The Frenet formulae of the spacelike curve $\alpha_2(t)$ can be written in the form

$$T'(t) = f_2(t) \times T(t), \quad N'(t) = f_2(t) \times N(t), \quad B'(t) = f_2(t) \times B(t)$$

by the vector $f_2(t)$ (Figure 4. 3).

Proof. It is sufficient to multiply the Frenet vectors by $f_2(t)$ vectorically in the Lorentz sense.

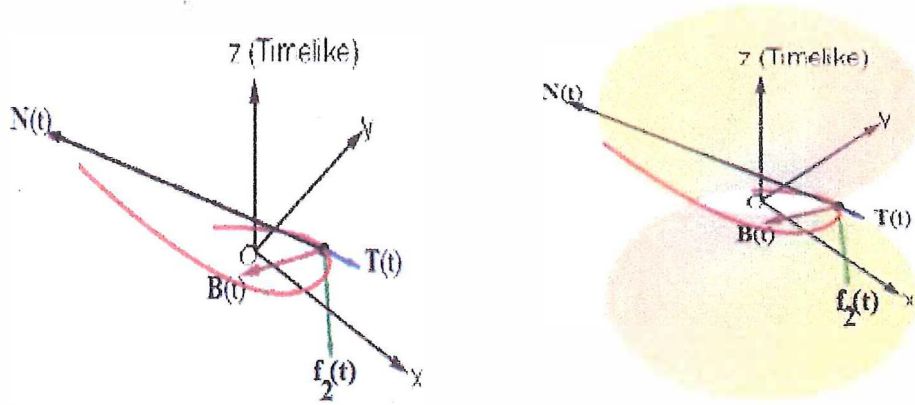


Fig.4.3: Frenet trihedron and the instantaneous rotation vector of the curve $\alpha_2(t)$

5 Lorentzian Viviani Curve III

In this chapter, we define the parametric equations and examine the geometry of the curve which is the intersection of the Lorentzian sphere of the radius a centered at the origin with hyperbolic cylinder with radius a centered at $(0, 0, a)$ at the Lorentz - Minkowski 3 - space.

A parameterization for this curve is given by

$$\alpha_3(t) = (a \sinh t, \pm a \sqrt{3 + 2 \cosh t}, a(1 + \cosh t)), t \in \mathbb{R} \quad (5.1)$$

where a is positive. The curve defined by (5.1) is called *Lorentziyen Viviani eğrisi III*. The curve is on upper sheed of the hyperbolic cylinder (Figure 5.1).

For this curve, we have

$$\nu_3(t) = \|\alpha_3'(t)\| = a \sqrt{\frac{4 \cosh t + \cosh 2t + 5}{2(3 + 2 \cosh t)}} \quad (5.2)$$

from

$$\alpha_3'(t) = \left(a \cosh t, \pm a \frac{\sinh t}{\sqrt{3 + 2 \cosh t}}, a \sinh t \right). \quad (5.3)$$

Since $\cosh t \geq 1$ and $\cosh 2t \geq 1$; by (5.2), $\alpha_3(t)$ is a spacelike curve and it is not parametrized by the arc length.

If the second component of the curve $\alpha_3(t)$ is taken as $+$, the red colored curve arc is obtained and if it is taken as $-$, the green curve arc is obtained, as seen in figure (5.1).

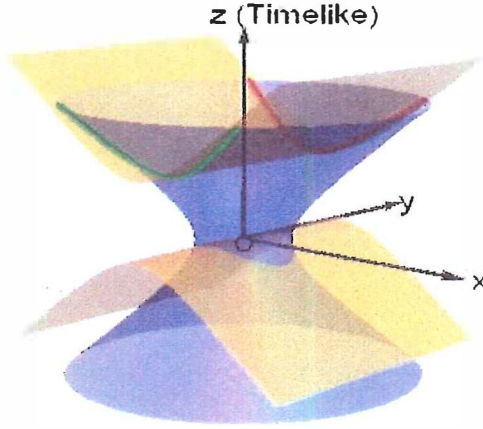


Fig. 5. 1: Lorentzian Viviani curve III, $\alpha = 1$.

Theorem 5.1 Let $\alpha_3 : I \subseteq \mathbb{R} \rightarrow S_1^2(\alpha)$ be regular Lorentzian Viviani curve III with arbitrary parameter t . In this case, the Frenet formulae of the Frenet frame $\{T, N, B\}$ are given by

$$\left. \begin{aligned} T'(t) &= \frac{\sqrt{2 \cosh^2 \frac{t}{2} (3 \cosh 3t + 22 \cosh 2t + 85 \cosh t + 90)}}{(4 \cosh t + \cosh 2t + 5) \sqrt{3 + 2 \cosh t}} N(t) \\ N'(t) &= \frac{\sqrt{2 \cosh^2 \frac{t}{2} (3 \cosh 3t + 22 \cosh 2t + 85 \cosh t + 90)}}{(4 \cosh t + \cosh 2t + 5) \sqrt{3 + 2 \cosh t}} T(t) \mp \\ &\quad \frac{\frac{3\sqrt{2}}{2} \sqrt{4 \cosh t + \cosh 2t + 5} \left(5 \sinh \frac{3t}{2} + \sinh \frac{5t}{2} \right) \operatorname{sech} \frac{t}{2}}{85 \cosh t + 22 \cosh 2t + 3 \cosh 3t + 90} B(t) \\ B'(t) &= \mp \frac{\frac{3\sqrt{2}}{2} \sqrt{4 \cosh t + \cosh 2t + 5} \left(5 \sinh \frac{3t}{2} + \sinh \frac{5t}{2} \right) \operatorname{sech} \frac{t}{2}}{85 \cosh t + 22 \cosh 2t + 3 \cosh 3t + 90} N(t) \end{aligned} \right\} \quad (5.4)$$

Proof. By (5.2) and (5.3) the unit tangent vector field is

$$T(t) = \frac{1}{\sqrt{\frac{4 \cosh t + \cosh 2t + 5}{2(3 + 2 \cosh t)}}} \left(\cosh t, \pm \frac{\sinh t}{\sqrt{3 + 2 \cosh t}}, \sinh t \right).$$

by a direct calculations, we have

$$\alpha_3''(t) = a \left(\sinh t, \pm \frac{\cosh t \sqrt{3 + 2 \cosh t} - \frac{\sinh^2 t}{\sqrt{3 + 2 \cosh t}}}{3 + 2 \cosh t}, \cosh t \right)$$

$$\alpha_3''(t) = a \left(\sinh t, \pm \frac{6 \cosh t + \cosh 2t + 3}{2(3 + 2 \cosh t)^{3/2}}, \cosh t \right),$$

$$\alpha_3'(t) \times \alpha_3''(t) = a^2 \begin{vmatrix} -e_1 & -e_2 & e_3 \\ \cosh t & \pm \frac{\sinh t}{\sqrt{3 + 2 \cosh t}} & \sinh t \\ \sinh t & \pm \frac{6 \cosh t + \cosh 2t + 3}{2(3 + 2 \cosh t)^{3/2}} & \cosh t \end{vmatrix}$$

$$= a^2 \left(\pm \frac{6 \cosh t \sinh t + \cosh 2t \sinh t + 3 \sinh t}{2(3 + 2 \cosh t)^{3/2}} \mp \frac{\sinh t \cosh t}{\sqrt{3 + 2 \cosh t}}, 1, \right.$$

$$\left. \pm \frac{6 \cosh^2 t + \cosh 2t \cosh t + 3 \cosh t}{2(3 + 2 \cosh t)^{3/2}} \mp \frac{\sinh^2 t}{\sqrt{3 + 2 \cosh t}} \right)$$

$$\alpha_3'(t) \times \alpha_3''(t) = a^2 \left(\mp \frac{\sinh^3 t}{(3+2 \cosh t)^{3/2}}, 1, \mp \frac{\cosh 3t - 9 \cosh t - 12}{4(3+2 \cosh t)^{3/2}} \right), \quad (5.5)$$

$$\|\alpha_3'(t) \times \alpha_3''(t)\| = a^2 \sqrt{\frac{\cosh^2 \frac{t}{2} (3 \cosh 3t + 22 \cosh 2t + 85 \cosh t + 90)}{2(3+2 \cosh t)^3}} \quad (5.6)$$

and

$$\langle \alpha_3'(t) \times \alpha_3''(t), \alpha_3'(t) \times \alpha_3''(t) \rangle = a^4 \left(\frac{\sinh^6 t}{(3+2 \cosh t)^3} + 1 - \frac{(\cosh 3t - 9 \cosh t - 12)^2}{16(3+2 \cosh t)^3} \right)$$

$$= \frac{a^4}{16(3+2 \cosh t)^3} (16 \sinh^6 t + 16(3+2 \cosh t)^3 - \cosh^2 3t - 81 \cosh^2 t - 144 + 18 \cosh 3t \cosh t + 24 \cosh 3t - 108 \cosh t)$$

$$= \frac{a^4 \cosh^2 \frac{t}{2} (3 \cosh 3t + 22 \cosh 2t + 85 \cosh t + 90)}{2(3+2 \cosh t)^3}.$$

Thus, the curvature function of the curve $\alpha_3(t)$ is given by

$$\kappa(t) = \frac{a^2 \sqrt{\frac{\cosh^2 \frac{t}{2} (3 \cosh 3t + 22 \cosh 2t + 85 \cosh t + 90)}{2(3+2 \cosh t)^3}}}{a^3 \left(\frac{4 \cosh t + \cosh 2t + 5}{2(3+2 \cosh t)} \right)^{3/2}}$$

or

$$\kappa(t) = \frac{2 \cosh \frac{t}{2} \sqrt{3 \cosh 3t + 22 \cosh 2t + 85 \cosh t + 90}}{a(4 \cosh t + \cosh 2t + 5)^{3/2}}. \quad (5.7)$$

By (5.5) and (5.6), unit binormal vector field is found in this form

$$B(t) = \frac{\sqrt{2}}{\cosh \frac{t}{2} \sqrt{3 \cosh 3t + 22 \cosh 2t + 85 \cosh t + 90}} \left(\pm \sinh^3 t, \right. \\ \left. -(3 + 2 \cosh t)^{3/2}, \pm \frac{1}{4} (\cosh 3t - 9 \cosh t - 12) \right).$$

The cross product $B(t) \times T(t)$ give principle normal vector field

$$N(t) = \frac{2\sqrt{3 + 2 \cosh t}}{\cosh \frac{t}{2} \sqrt{4 \cosh t + \cosh 2t + 5} \sqrt{3 \cosh 3t + 22 \cosh 2t + 85 \cosh t + 90}} \cdot$$

$$\begin{vmatrix} -e_1 & -e_2 & e_3 \\ \pm \sinh^3 t & -(3 + 2 \cosh t)^{3/2} & \pm \frac{1}{4} (\cosh 3t - 9 \cosh t - 12) \\ \cosh t & \pm \frac{\sinh t}{\sqrt{3 + 2 \cosh t}} & \sinh t \end{vmatrix}$$

or

$$N(t) = \frac{1}{\cosh \frac{t}{2} \sqrt{4 \cosh t + \cosh 2t + 5} \sqrt{3 \cosh 3t + 22 \cosh 2t + 85 \cosh t + 90}},$$

$$\left(\frac{1}{4} (56 \sinh t + 38 \sinh 2t + 8 \sinh 3t + \sinh 4t), \pm \sqrt{3 + 2 \cosh t} (6 \cosh t + \cosh 2t + 3), \right.$$

$$\left. \pm \frac{1}{4} (96 \cosh t + 44 \cosh 2t + 8 \cosh 3t + \cosh 4t + 51) \right).$$

Since $\langle N(t), N(t) \rangle = -1$, the vector field $N(t)$ is timelike. Third derivative of the curve is found in the form

$$\alpha_3'''(t) = a(\cosh t,$$

$$\pm \frac{(6 \sinh t + 2 \sinh 2t) 2 (3 + 2 \cosh t)^{3/2} - (\cosh 2t + 6 \cosh t + 3) 6 \sinh t \sqrt{3 + 2 \cosh t}}{4 (3 + 2 \cosh t)^3}, \sinh t)$$

$$= a\left(\cosh t, \pm a \frac{6 \sinh t + 2 \sinh 2t}{2 (3 + 2 \cosh t)^{3/2}} - \frac{3 \sinh t (\cosh 2t + 6 \cosh t + 3)}{2 (3 + 2 \cosh t)^{5/2}}, \sinh t\right)$$

or if the necessary simplifications are made in the second component of this vector,

$$\alpha_3'''(t) = a\left(\cosh t, \pm \frac{\sinh(3t) + 6 \sinh(2t) + 25 \sinh t}{4 (3 + 2 \cosh t)^{5/2}}, \sinh t\right).$$

The determinant of the first three derivatives and torsion function are given by

$$\det(\alpha_3'(t), \alpha_3''(t), \alpha_3'''(t)) = -\langle \alpha_3'(t) \times \alpha_3''(t), \alpha_3'''(t) \rangle$$

$$\begin{aligned}
&= - \left\langle a^2 \left(\mp \frac{\sinh^3 t}{(3+2 \cosh t)^{3/2}}, 1, \mp \frac{\cosh 3t - 9 \cosh t - 12}{4(3+2 \cosh t)^{3/2}} \right), \right. \\
&\quad \left. \left(\cosh t, \pm \frac{\sinh 3t + 6 \sinh 2t + 25 \sinh t}{4(3+2 \cosh t)^{5/2}}, \sinh t \right) \right\rangle \\
&= a^3 \left(\pm \frac{\cosh t \sinh^3 t}{(3+2 \cosh t)^{3/2}} \mp \frac{\sinh 3t + 6 \sinh 2t + 25 \sinh t}{4(3+2 \cosh t)^{5/2}} \mp \right. \\
&\quad \left. \frac{\sinh t \cosh 3t - 9 \sinh t \cosh t - 12 \sinh t}{4(3+2 \cosh t)^{3/2}} \right) \\
&= \mp a^3 \frac{1}{4(3+2 \cosh t)^{5/2}} \left(-4(3+2 \cosh t) \cosh t \sinh^3 t + \sinh 3t + \sinh 2t \right. \\
&\quad \left. + 25 \sinh t + (3+2 \cosh t)(\sinh t \cosh 3t - 9 \sinh t \cosh t - 12 \sinh t) \right) \\
&= \mp a^3 \frac{3 \sinh t (3 + 6 \cosh t + \cosh 2t)}{2(3+2 \cosh t)^{5/2}}
\end{aligned}$$

and

$$\begin{aligned}
\tau(t) &= \frac{\mp a^3 \frac{3 \sinh t (3 + 6 \cosh t + \cosh 2t)}{2(3+2 \cosh t)^{5/2}}}{a^4 \frac{\cosh^2 \frac{t}{2} (3 \cosh 3t + 22 \cosh 2t + 85 \cosh t + 90)}{2(3+2 \cosh t)^3}} \\
&= \mp \frac{6\sqrt{3+2 \cosh t} (3 + 6 \cosh t + \cosh 2t) \tanh \frac{t}{2}}{a(3 \cosh 3t + 22 \cosh 2t + 85 \cosh t + 90)}
\end{aligned}$$

$$\tau(t) = \mp \frac{3 \operatorname{sech} \frac{t}{2} \sqrt{3+2 \cosh t} \left(5 \sinh \frac{3t}{2} + \sinh \frac{5t}{2} \right)}{a(85 \cosh t + 22 \cosh 2t + 3 \cosh 3t + 90)} \quad (5.8)$$

respectively. At the minimum point $\alpha_3(0) = (0, \pm a\sqrt{5}, 2a)$ for $t = 0$, torsion is zero (Figure 5.2).

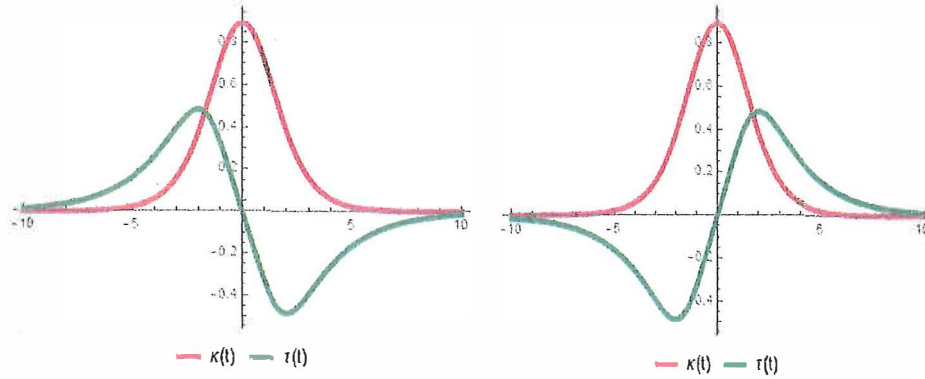


Fig. 5. 2: Curvature and torsion functions of the Lorentzian Viviani curve III, $a=1$.

Corollary 5. 2 The Frenet instantaneous rotation vector of the frame $\{T(t), N(t), B(t)\}$ of the curve $\alpha_3(t)$ can be written in the form

$$f_3(t) = \pm \frac{\frac{3\sqrt{2}}{2} \operatorname{sech} \frac{t}{2} \sqrt{4 \cosh t + \cosh 2t + 5} \left(5 \sinh \frac{3t}{2} + \sinh \frac{5t}{2} \right)}{85 \cosh t + 22 \cosh 2t + 3 \cosh 3t + 90} T(t) + \frac{\sqrt{2 \cosh^2 \frac{t}{2} (3 \cosh 3t + 22 \cosh 2t + 85 \cosh t + 90)}}{\sqrt{3+2 \cosh t} (4 \cosh t + \cosh 2t + 5)} B(t) \quad (5.9)$$

(Figure 5. 3).

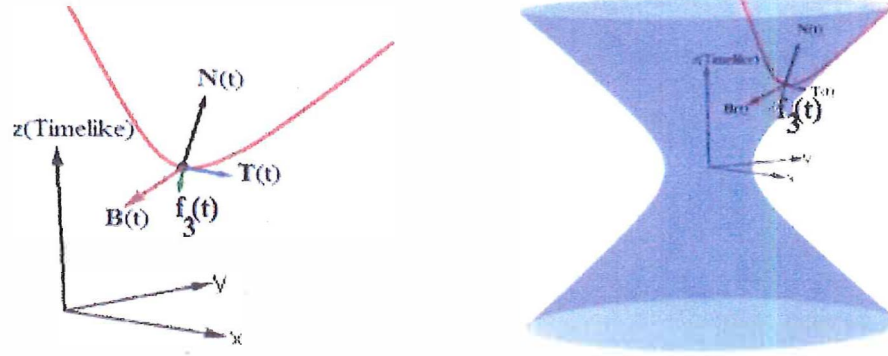


Fig. 5. 3: Frenet trihedron and instantaneous rotation vector of the curve $\alpha_3(t)$

Corollary 5.3 The Frenet formulas of the timelike arc of the curve $\alpha_3(t)$ can be written in the form

$$\mathbf{T}'(t) = f_3(t) \times \mathbf{T}(t), \quad \mathbf{N}'(t) = f_3(t) \times \mathbf{N}(t), \quad \mathbf{B}'(t) = f_3(t) \times \mathbf{B}(t)$$

by the vector $f_3(t)$.

Proof. It is sufficient to multiply the Frenet vectors by $f_3(t)$ vectorically in the Lorentz sense.

6 Lorentziyen Viviani Curve IV

In this chapter, we define the parametric equations and examine the geometry of the curve which is the intersection of the Lorentzian sphere of the radius a centered at the origin with hyperbolic cylinder with radius a centered at $(0, 0, a)$ at the Lorentz - Minkowski 3 - space.

The parametric equations of the lower sheet of the hyperbolic cylinder are given by

$$x = a \sinh t, \quad z = a - a \cosh t. \quad (6.1)$$

If these equations are written in equation of the sphere, it seen that

$$y = \pm a\sqrt{3-2\cosh t}. \quad (6.2)$$

By (6.1) and (6.2), the parametrization for this curve is given by

$$\alpha_4(t) = (a \sinh t, \pm a\sqrt{3-2\cosh t}, a(1-\cosh t)), \quad (6.3)$$

where $t \in \left[-\cosh^{-1} \frac{3}{2}, \cosh^{-1} \frac{3}{2}\right]$. The curve defined by (6.3) is called *Lorentzian Viviani curve IV*. The curve is on lower sheed of the hyperbolic cylinder (Figure 6.2), by (6.4), it is not parametrized by arc lenght and is spacelike curve as seen in figure 6.1.

For this curve, we have

$$v_4(t) = \|\alpha_4'(t)\| = a\sqrt{\frac{-4\cosh t + \cosh 2t + 5}{2(3-2\cosh t)}} \quad (6.4)$$

from

$$\alpha_4'(t) = \left(a \cosh t, \mp a \frac{\sinh t}{\sqrt{3-2\cosh t}}, -a \sinh t \right). \quad (6.5)$$

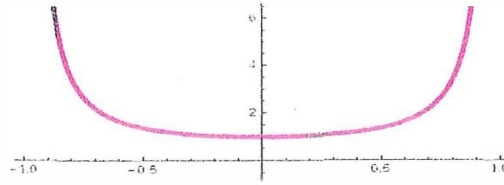


Fig. 6. 1: The graphic of the function $\langle \alpha_4'(t), \alpha_4'(t) \rangle$; $t \in \left[-\cosh^{-1} \frac{3}{2}, \cosh^{-1} \frac{3}{2}\right]$, $a=1$.

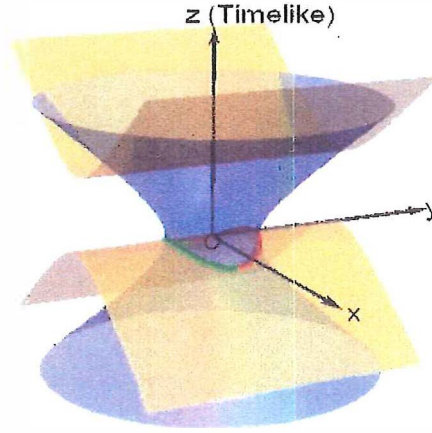
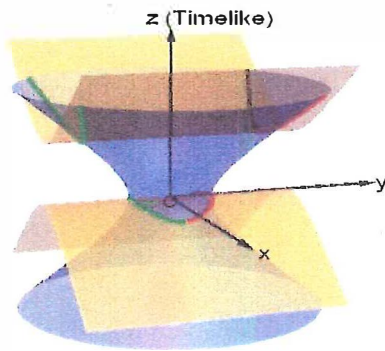
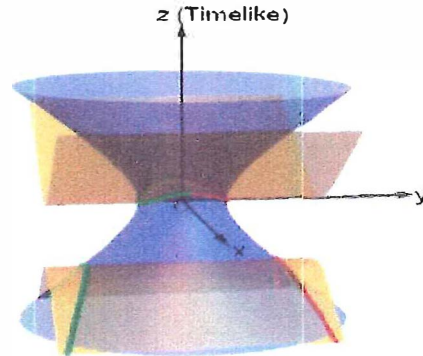


Fig. 6. 2: Lorentzian Viviani curve IV , $a = 1$.

If the second component of the curve $\alpha_4(t)$ given by (6.3) is taken as +, the red curve arc on the lower sheet of the cylinder is found, and if it is taken as -, the green curve arc is found, as seen in figure 6.2. If the Cartesian equation of the hyperbolic cylinder is taken as $-x^2 + (z+a)^2 = a^2$, the symmetry of the curves with respect to the XY - plane is obtained, as seen in figure 6.3 (b).



(a)



(b)

Fig. 6. 3: Lorentzian Viviani curve III and IV

Theorem 6.1 Let $\alpha_4 : I \subseteq \mathbb{R} \rightarrow S_1^2(\alpha)$ be regular Lorentzian Viviani curve IV given by (6.3). In this case, the Frenet equations of the Frenet frame $\{T, N, B\}$ of the curve $\alpha_4(t)$ are given by

$$\left. \begin{aligned} T'(t) &= \frac{\sqrt{2 \sinh^2 \frac{t}{2} (-3 \cosh 3t + 22 \cosh 2t - 85 \cosh t + 90)}}{(-4 \cosh t + \cosh 2t + 5) \sqrt{3 - 2 \cosh t}} N(t) \\ N'(t) &= - \frac{\sqrt{2 \sinh^2 \frac{t}{2} (-3 \cosh 3t + 22 \cosh 2t - 85 \cosh t + 90)}}{(-4 \cosh t + \cosh 2t + 5) \sqrt{3 - 2 \cosh t}} T(t) \pm \\ &\quad \frac{\frac{3\sqrt{2}}{2} \sqrt{-4 \cosh t + \cosh 2t + 5} \left(-5 \cosh \frac{3t}{2} + \cosh \frac{5t}{2} \right) \operatorname{csch} \frac{t}{2}}{-3 \cosh 3t + 22 \cosh 2t - 85 \cosh t + 90} B(t) \\ B'(t) &= \pm \frac{\frac{3\sqrt{2}}{2} \sqrt{-4 \cosh t + \cosh 2t + 5} \left(-5 \cosh \frac{3t}{2} + \cosh \frac{5t}{2} \right) \operatorname{csch} \frac{t}{2}}{-3 \cosh 3t + 22 \cosh 2t - 85 \cosh t + 90} N(t). \end{aligned} \right\} \quad (6.6)$$

Proof. By (6.4) and (6.5) the unit tangent vector field is

$$T(t) = \frac{1}{\sqrt{\frac{-4 \cosh t + \cosh 2t + 5}{2(3 - 2 \cosh t)}}} \left(\cosh t, \mp \frac{\sinh t}{\sqrt{3 - 2 \cosh t}}, -\sinh t \right).$$

By direct calculations, we have

$$\alpha_4''(t) = \alpha \left(\sinh t, \mp \frac{\cosh t \sqrt{3 - 2 \cosh t} + \frac{\sinh t}{\sqrt{3 - 2 \cosh t}}}{3 - 2 \cosh t}, -\cosh t \right)$$

$$\alpha_4''(t) = \alpha \left(\sinh t, \mp \frac{-\cosh 2t + 6 \cosh t - 3}{2(3 - 2 \cosh t)^{3/2}}, -\cosh t \right),$$

$$\alpha_4'(t) \times \alpha_4''(t) = \alpha^2 \begin{vmatrix} -e_1 & -e_2 & e_3 \\ \cosh t & \mp \frac{\sinh t}{\sqrt{3 - 2 \cosh t}} & -\sinh t \\ \sinh t & \mp \frac{6 \cosh t - \cosh 2t - 3}{2(3 - 2 \cosh t)^{3/2}} & -\cosh t \end{vmatrix}$$

$$= \alpha^2 \left(\mp \frac{-6 \cosh t \sinh t + \cosh 2t \sinh t + 3 \sinh t}{2(3 - 2 \cosh t)^{3/2}} \mp \frac{\sinh t \cosh t}{\sqrt{3 - 2 \cosh t}}, -1, \right. \\ \left. \mp \frac{6 \cosh^2 t - \cosh 2t \cosh t - 3 \cosh t}{2(3 - 2 \cosh t)^{3/2}} \pm \frac{\sinh^2 t}{\sqrt{3 - 2 \cosh t}} \right) \\ = \alpha^2 \left(\pm \frac{\sinh^3 t}{(3 - 2 \cosh t)^{3/2}}, -1, \mp \frac{\cosh 3t - 9 \cosh t + 12}{4(3 - 2 \cosh t)^{3/2}} \right),$$

$$\|\alpha_4'(t) \times \alpha_4''(t)\| = \alpha^2 \sqrt{\left| \frac{\sinh^2 \frac{t}{2} (3 \cosh 3t - 22 \cosh 2t + 85 \cosh t - 90)}{2(3 - 2 \cosh t)^3} \right|} \\ = \alpha^2 \sqrt{\frac{\sinh^2 \frac{t}{2} (-3 \cosh 3t + 22 \cosh 2t - 85 \cosh t + 90)}{2(3 - 2 \cosh t)^3}},$$

and

$$\langle \alpha_4'(t) \times \alpha_4''(t), \alpha_4'(t) \times \alpha_4''(t) \rangle = \frac{\alpha^4 \sinh^2 \frac{t}{2} (3 \cosh 3t - 22 \cosh 2t + 85 \cosh t - 90)}{2(3 - 2 \cosh t)^3}$$

(Figure 6. 4).

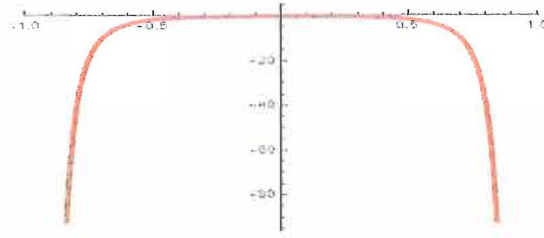


Fig. 6.4: The graphic of function $\langle \alpha_4'(t) \times \alpha_4''(t), \alpha_4'(t) \times \alpha_4''(t) \rangle$,

$$t \in \left[-\cosh^{-1} \frac{3}{2}, \cosh^{-1} \frac{3}{2} \right], a=1.$$

Accordingly, null frame is defined at $t=0$ ($\alpha_4(0)=(0, \pm 1, 0)$) and for $t \neq 0$, the unit principle normal and unit timelike binormal vector fields are found in the forms

$$N(t) = \frac{2\sqrt{3-2\cosh t}}{\sqrt{-4\cosh t + \cosh 2t + 5} \sqrt{\sinh^2 \frac{t}{2} (-3\cosh 3t + 22\cosh 2t - 85\cosh t + 90)}} \begin{vmatrix} -e_1 & -e_2 & e_3 \\ \cosh t & \mp \frac{\sinh t}{\sqrt{3-2\cosh t}} & -\sinh t \\ \pm \sinh^3 t & -(3-2\cosh t)^{3/2} & \mp \frac{1}{4}(\cosh 3t - 9\cosh t + 12) \end{vmatrix}$$

$$N(t) = \frac{1}{\sqrt{-4 \cosh t + \cosh 2t + 5} \sqrt{\sinh^2 \frac{t}{2} (-3 \cosh 3t + 22 \cosh 2t - 85 \cosh t + 90)}} \\ \left(\frac{-56 \sinh t + 38 \sinh 2t - 8 \sinh 3t + \sinh 4t}{4}, \mp \sqrt{3 - 2 \cosh t} (3 - 6 \cosh t + \cosh 2t), \right. \\ \left. -\frac{1}{4} (-96 \cosh t + 44 \cosh 2t - 8 \cosh 3t + \cosh 4t + 51) \right),$$

and

$$B(t) = \frac{\sqrt{2} (3 - 2 \cosh t)^{3/2}}{\sqrt{\sinh^2 \frac{t}{2} (-3 \cosh 3t + 22 \cosh 2t - 85 \cosh t + 90)}} \left(\pm \frac{\sinh^3 t}{(3 - 2 \cosh t)^{3/2}}, \right. \\ \left. -1, \mp \frac{\cosh 3t - 9 \cosh t + 12}{4(3 - 2 \cosh t)^{3/2}} \right) \\ B(t) = \frac{\sqrt{2}}{\sqrt{\sinh^2 \frac{t}{2} (-3 \cosh 3t + 22 \cosh 2t - 85 \cosh t + 90)}} \left(\pm \sinh^3 t, \right. \\ \left. -(3 - 2 \cosh t)^{3/2}, \mp \frac{1}{4} (\cosh 3t - 9 \cosh t + 12) \right),$$

respectively.

Therefore, the curvature function is

$$\kappa(t) = \frac{a^2 \sqrt{\sinh^2 \frac{t}{2} (-3 \cosh 3t + 22 \cosh 2t - 85 \cosh t + 90)}}{2(3 - 2 \cosh t)^3} \cdot \frac{1}{a^3 \left(\frac{-4 \cosh t + \cosh 2t + 5}{2(3 - 2 \cosh t)} \right)^{3/2}}$$

$$\kappa(t) = \frac{2 \sqrt{\sinh^2 \frac{t}{2} (-3 \cosh 3t + 22 \cosh 2t - 85 \cosh t + 90)}}{a(-4 \cosh t + \cosh 2t + 5)^{3/2}}, \quad (6.7)$$

Let us the third derivative of the curve:

$$\alpha_4'''(t) = a \left(\cosh t, \right. \\ \left. \mp \frac{(6 \sinh t - 2 \sinh 2t) 2(3 - 2 \cosh t)^{3/2} - (-\cosh 2t + 6 \cosh t - 3)(-6 \sinh t) \sqrt{3 - 2 \cosh t}}{4(3 - 2 \cosh t)^3}, \right. \\ \left. -\sinh t \right)$$

$$= a \left(\cosh t, \mp \frac{6 \sinh t - 2 \sinh 2t}{2(3 - 2 \cosh t)^{3/2}} + \frac{3 \sinh t (-\cosh 2t + 6 \cosh t - 3)}{2(3 - 2 \cosh t)^{5/2}}, -\sinh t \right)$$

$$= a \left(\cosh t, \pm \frac{\sinh 3t - 6 \sinh 2t + 25 \sinh t}{4(3 - 2 \cosh t)^{5/2}}, -\sinh t \right).$$

The determinant of the first three derivatives and torsion function are given by

$$\det(\alpha_4'(t), \alpha_4''(t), \alpha_4'''(t)) = -\langle \alpha_4'(t) \times \alpha_4''(t), \alpha_4'''(t) \rangle$$

$$\det(\alpha_4'(t), \alpha_4''(t), \alpha_4'''(t)) = - \left\langle a^2 \left(\pm \frac{\sinh^3 t}{(3-2 \cosh t)^{3/2}}, -1, \mp \frac{\cosh 3t - 9 \cosh t + 12}{4(3-2 \cosh t)^{3/2}} \right), \right. \\ \left. a \left(\cosh t, \pm \frac{\sinh 3t - 6 \sinh 2t + 25 \sinh t}{4(3-2 \cosh t)^{5/2}}, -\sinh t \right) \right\rangle \\ \det(\alpha_4'(t), \alpha_4''(t), \alpha_4'''(t)) = \mp a^3 \frac{3 \sinh t (3 - 6 \cosh t + \cosh 2t)}{2(3-2 \cosh t)^{5/2}}.$$

Thus, by (2.9), the torsion function is given by

$$\tau(t) = \frac{\mp a^3 \frac{3 \sinh t (3 - 6 \cosh t + \cosh 2t)}{2(3-2 \cosh t)^{5/2}}}{a^4 \frac{\sinh^2 \frac{t}{2} (-3 \cosh 3t + 22 \cosh 2t - 85 \cosh t + 90)}{2(3-2 \cosh t)^3}}$$

or

$$\tau(t) = \pm \frac{3 \operatorname{csch} \frac{t}{2} \sqrt{3-2 \cosh t} \left(-5 \cosh \frac{3t}{2} + \cosh \frac{5t}{2} \right)}{a(-3 \cosh 3t + 22 \cosh 2t - 85 \cosh t + 90)}. \quad (6.8)$$

For $t = \pm \cosh^{-1} \frac{3}{2}$, the torsion at the points

$\alpha_4 \left(\pm \cosh^{-1} \frac{3}{2} \right) = a \left(\pm \frac{\sqrt{5}}{2}, 0, -\frac{1}{2} \right)$ is zero (Figure 6. 5) and is undefined for $t = 0$. The invariants (6.7) and (6.8) satisfies the Frenet derivative formulas (6.6).

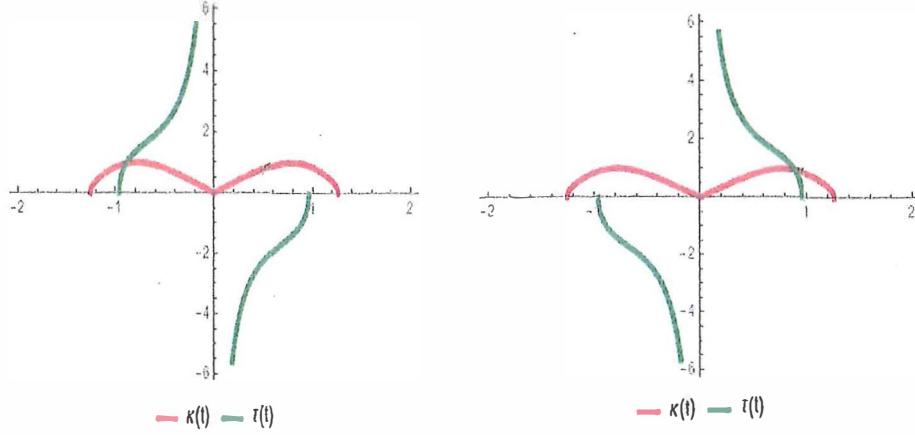


Fig. 6. 5: Curvature and torsion functions of the Lorentzian Viviani Curve IV, $t \in \left[-\cosh^{-1} \frac{3}{2}, \cosh^{-1} \frac{3}{2}\right]$, $a = 1$.

Corollary 6. 2 By (6.7) and (6.8), the Frenet instantaneous rotation vector of the Frenet frame $\{T, N, B\}$ of the Lorentzian Viviani curve IV can be written in the form

$$f_{\omega}(t) = \pm \frac{\frac{3\sqrt{2}}{2} \operatorname{csch} \frac{t}{2} \sqrt{-4 \cosh t + \cosh 2t + 5} \left(-5 \cosh \frac{3t}{2} + \cosh \frac{5t}{2}\right)}{-3 \cosh 3t + 22 \cosh 2t - 85 \cosh t + 90} T(t) - \frac{\sqrt{2 \sinh^2 \frac{t}{2} (-3 \cosh 3t + 22 \cosh 2t - 85 \cosh t + 90)}}{\sqrt{3 - 2 \cosh t} (-4 \cosh t + \cosh 2t + 5)} B(t) \quad (6.9)$$

(Figure 6. 6).

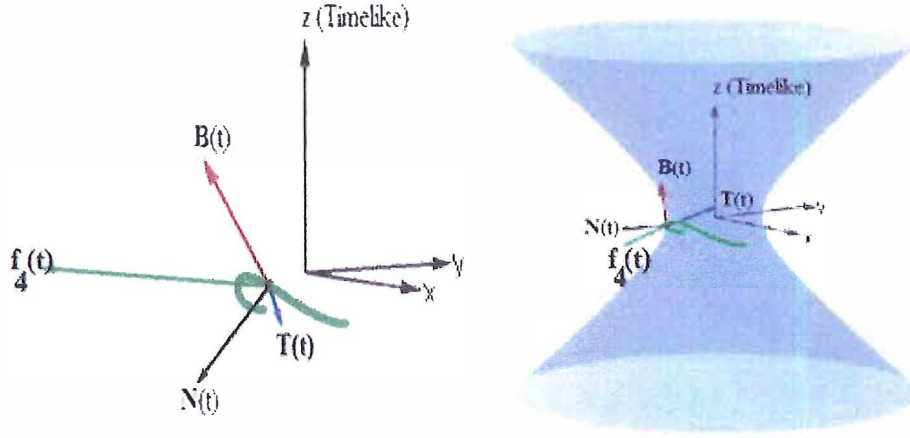


Fig. 6. 6: The Frenet frame and the instantaneous rotation vector of the Lorentzian Viviani curve IV.

Corollary 6. 3 The Frenet formulas of the spacelike curve $\alpha_4(t)$ are given by

$$T'(t) = f_4(t) \times T(t), \quad N'(t) = f_4(t) \times N(t), \quad B'(t) = f_4(t) \times B(t)$$

by the vector $f_4(t)$. (Figure 6. 6)

Proof. It is sufficient to multiply the Frenet vectors by $f_4(t)$ vectorically in the Lorentz sense.

References

- [1] Svinin, M. Planning of smooth motions for a ball-plate system with limited contact area. Kyushu University, International conference on robotics and automation, May 2008, DOI:10.1109/ROBOT.2008.4543366.Source:DBLP.
- [2] Watt, A. 3D Computers Graphics. Third Edition, Addison Wesley, Pearson Education Limited, England, 2000.

- [3] Yamaguchi, F. Curves and Surfaces in ComputerAided Geometric Design. Springer –Verlag Berlin Heidelberg New York London Paris Tokyo.
- [4] Caddeo, R., Montaldo, S., Piu, P. The Mobius Strip and the Viviani's Windows. The Mathematical Intelligencer, Volume 23, September, 35-39 pp., 2001.
- [5] Graefe, E-M., Korsch, H.J., Strzys, M.P. Bose-Hubbard dimers, Viviani's Windows and pendulum Dynamics. arXiv, 2014.
- [6] Goemans, W. Clelia Curves in Euclidean and Minkowski 3-space. AMS Sectional Meeting AMS Special Session on Recent Advances in the Geometry of Submanifolds, Dedicated to the Memory of Franki Dillen (1963-2013), Joint work with Ignace Van de Woestyne, K.U. Leuven, March 15th, 2015.
- [7] Barros, M., Ferrandez, A., Lucas, P. and Merono, A. M. General helices in the three-dimensional Lorentzian space forms. Rocky Mountain J. Math.31(2), 373-388, 2001.
- [8] Barros, M. General helices and a theorem of Lancret. Proc. Am. Math. Soc., 125 (1997), 1503-1509.
- [9] Ferrández, A., Giménez, A., P. Lucas, P. Null generalized helices in Lorentz Minkowski spaces. Phys. A, 35 (39), 8243-8251, 2002.
- [10] Walrave, J. Curves and surfaces in Minkowski space. Katholieke Universiteit Leuven (Belgium), 1995, Thesis (Ph.D.).
- [11] Uğurlu, H. H. , Kabadayı, H., Yaşar, G. Timelike Helices on the Lorentzian Sphere $S_1^2(r)$. 16 th International Geometry Symposium, Manisa, Turkey, 2018.
- [12] Uğurlu, H. H., Bulut, V., Aydınalp, M. The helix curves on the spheres of Lorentz-Minkowski space $\mathbb{R}_1^3$. Hadronic Journal, Vol.44, No:3, p.227, September 2021.
- [13] Uğurlu, H. H., Bulut, V. The Spirals on The Oblate and Prolate Spheroids of Lorentz - Minkowski 3-Space $\mathbb{R}_1^3$. Facta Universitatis, Series: Mathematics and Informatics, Vol.38, No.2, 285-299 pp, 2023.
- [14] Taner, T. $\mathbb{R}_1^3$ Lorentz - Minkowski 3-Uzayında Viviani Eğrilerinin Geometrisi. Celal Bayar Üniversitesi, Manisa, Türkiye, 2022, Doktora Tezi.
- [15] Uğurlu, H.H., Çalışkan, A. Darboux Ani Dönme Vektörleri ile Spacelike ve Timelike Yüzeyler Geometrisi. Celal Bayar Üniversitesi Yayınları, Yayın No:0006, 170 s., Manisa, Türkiye, 2012.

- [16] O'Neill, B. Semi-Riemannian Geometry with Applications to Relativity. Academic Press, London, 1983.
- [17] Couto, I.T., & Lymberopoulos, A. Introduction to Lorentz Geometry: Curves and Surfaces (1st ed.). Chapman and Hall/CRC., 351 p., 2021.
- [18] Lopez, R. Differential Geometry of Curves and Surfaces in Lorentz-Minkowski Space. International Electronic Journal of Geometry, Volume 7, No.1, 44-107 pp, 2014.
- [19] Kühnel, W. Differential Geometry Curves –Surfaces –Manifolds. Second Edition, Translated by Bruce Hunt, Student Mathematical Library, Volume 16, AMS, 380p., U.S.A., 2006.
- [20] Yoon, D.W. On The Gauss Map Of Translation Surfaces in Minkowski 3-Space. Taiwanese Journal of Mathematics Vol.6, No.3, 389-398pp., September 2002.
- [21] Aydınalp, M., Uğurlu, H.H. The causal spirals on the spheres of Lorentz-Minkowski 3-space $\mathbb{R}_1^3$. Algebras Groups and Geometries, 37, 81-101, 2021.
- [22] Gray, A. Modern Differential Geometry of Curves and Surfaces with Mathematica. Third Edition by Elsa Abbena and Simon Salamon, 996 p., December 2005.
- [23] Galinski, C., Zibikowski, R. Insect-like flapping wing mechanism based on a double spherical Scotch yoke. Journal of The Royal Society Interface, July 2005.
- [24] Graves, L. K. Codimension one isometric immersions between Lorentz spaces. Trans. of The Amer. Math. Soc., 252 (1979), 367-392.
- [25] Liu, H. Curves in the lightlike cone. Contributions to Algebra and Geometry. volume 45(2004), No.1, 291-303.
- [26] O'Neill, B. Elementary Differential Geometry. Revised second edition, Elsevier, London, 2006.
- [27] Carmo, M. do. Differential Geometry of Curves and Surfaces. Prentice-Hall, Saddle River, New Jersey, Brazil, 511 p., 1976.
- [28] Breuer, S. and Gottlieb, D. Explicit characterizations of spherical curves. Proc. Amer. Math. Soc. 27 (1971), 126-127.
- [29] Wong, Y. On an explicit characterization of spherical curves. Proc. Amer. Math. Soc. 34(1972), 239-242.
- [30] Scofield, P. D. Curves of constant precession. Amer. Math. Mon., 102 (1995), 531-537.

A VARIATIONAL FORMULATION FOR MODELING A
BOSE-EINSTEIN CONDENSATION IN AN
EULER-BERNOULLIAN CONTEXT

Fabio Silva Botelho

Department of Mathematics
Federal University of Santa Catarina – UFSC
Florianópolis, SC, Brazil
fabio.botelho@ufsc.br

Received July 31, 2025

Revised September 22, 2025

Abstract

This article develops a variational formulation for modeling a Bose-Einstein condensation process. The results are based on standard tools of calculus of variations and optimization theory. Moreover, we emphasize the context here addressed is essentially an Euler-Bernoullian one.

AMS Classification 35Q70, 35Q40, 76A02

Keywords. Euler system, Bose-Einstein condensate, strong variational formulations, Lagrange multipliers, optimization.

1 Introduction

The first part of this article develops a variational formulation for modeling a Bose-Einstein condensation in an Euler-Bernoullian context.

Such results are based on tools of calculus of variations and related Lagrange multipliers approach. It is worth emphasizing, in the Bose-Einstein condensation model here presented, a particle system corresponding to a fluid phase a , as submitted to a very low temperature (close to absolute zero), partially condensate into a phase b , known as the Bose-Einstein condensate. For such a condensed phase b , the particle system presents a single eigenvalue energy level, which depends only on the time t (through a temperature dependence) but does not depend on the macroscopic spatial variable y . In summary, the condensed phase b behaves somewhat like a "single particle field", with a unique quantum energy level.

Hence, concerning such scenarios, the particle mass constraint functionals for the phases a e b stand for

$$J_{Aux_1^a} = \int_0^{t_f} \int_{\Omega} E_1^a(y, t) \left(\int_{\Omega} (|\phi_a(x, y, t)|^2 dx - m_p^a) \right) dy dt, \quad (1)$$

where for such a phase a , the energy level $E_1^a(y, t)$ depends on y and t , and

$$J_{Aux_1^b} = \int_0^{t_f} E_1^b(t) \left(\int_{\Omega} (|\phi_b(x, t)|^2 dx - m_p^b) \right) dt, \quad (2)$$

where for such a phase b , the energy level $E_1^b(t)$ does not depend on y , that is, depends only on t , qualitatively behaving as a single energy state at each t .

Furthermore, we highlight the two main fields for such a system are a position one and a velocity one, which we specify below in the next lines.

Similar and related models in physics are addressed in [1, 2].

As standard references in theoretical fluid mechanics we would cite [3, 4, 5, 6, 7]. For related numerical methods we would mention [8, 10, 11]. Concerning other similar models, we would cite [12, 13, 14, 15, 16, 17].

Finally, details on the Sobolev spaces involved may be found in [18].

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and simply connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

Generically, we consider a fluid motion initially modeled as a particle system one, through the field of position $\mathbf{r} : \Omega \times [0, t_f] \rightarrow \mathbb{R}^3$ and field of velocities $\mathbf{v}$ where $\mathbf{v}(\mathbf{r}(x, t), t)$ stands for the velocity at the position $\mathbf{r}(x, t)$ and time $t \in [0, t_f]$.

Moreover, the gravity field is denoted by

$$\mathbf{g} = -g\mathbf{k},$$

related to the $\mathbb{R}^3$ basis

$$\{\mathbf{i} = (1, 0, 0), \mathbf{j} = (0, 1, 0), \mathbf{k} = (0, 0, 1)\}.$$

2 The mathematical model

In this section, we develop in details the concerning variational formulation for a compressible case.

We emphasize such a set $\Omega \subset \mathbb{R}^3$ stands for a control volume, in which along a time interval $[0, t_f]$, where $t \in [0, t_f]$ denotes time, a fluid particle system develops a compressible motion.

For a particle system corresponding to a phase a , we start by defining the following density field,

$$\hat{\rho}_a(x, y, t) = \left(|\phi_a(x, y, t)|^2 \frac{|\phi_a^M(y, t)|^2}{m_p^a} \right) \quad (3)$$

Here m_p^a denotes the mass of a single particle type a .

The macroscopic fluid density for such phase a , is defined by

$$\rho_a(y, t) = |\phi_a^M(y, t)|^2.$$

We recall that in the Bose-Einstein condensation, the condensed phase b has a unique energy level, which depends only on t but does not depend on the macroscopic variable y .

For such a particle system corresponding to the condensed phase b , we start by defining the following density field,

$$\hat{\rho}_b(x, y, t) = \left(|\phi_b(x, t)|^2 \frac{|\phi_b^M(y, t)|^2}{m_p^b} \right) \quad (4)$$

Here m_p^b denotes the mass of a single particle type b .

The macroscopic fluid density for such phase b , is defined by

$$\rho_b(y, t) = |\phi_b^M(y, t)|^2.$$

It is worth highlighting that generically, we denote

$$L^2(\Omega \times [0, t_f]) = \left\{ h : \Omega \times [0, t_f] \rightarrow \mathbb{R} \text{ measurable} : \int_0^{t_f} \int_{\Omega} |h(x, t)|^2 dx dt < +\infty \right\},$$

$$\begin{aligned} L^2(\Omega \times [0, t_f]; \mathbb{R}^3) &= \{ h = (h_1, h_2, h_3) : \Omega \times [0, t_f] \rightarrow \mathbb{R}^3 \\ & : h_k \in L^2(\Omega \times [0, t_f]), \forall k \in \{1, 2, 3\} \}, \end{aligned} \quad (5)$$

$$\begin{aligned} W^{1,2}(\Omega \times [0, t_f]) &= \{ h : \Omega \times [0, t_f] \rightarrow \mathbb{R} : h \in L^2(\Omega \times [0, t_f]) \\ & \text{and } h_t, h_{x_j} \in L^2(\Omega \times [0, t_f]), \forall j \in \{1, 2, 3\} \}, \end{aligned} \quad (6)$$

where the partial derivatives h_t, h_{x_j} are understood in a distributional sense, $\forall j \in \{1, 2, 3\}$.

Moreover,

$$\begin{aligned} W^{1,2}(\Omega \times [0, t_f]; \mathbb{R}^3) &= \{ h = (h_1, h_2, h_3) : \Omega \times [0, t_f] \rightarrow \mathbb{R}^3 \\ & : h_k \in W^{1,2}(\Omega \times [0, t_f]), \forall k \in \{1, 2, 3\} \} \end{aligned} \quad (7)$$

Similar notations are valid for other sets and other related complex and vectorial cases.

With such notations in mind, we define the concerning function spaces

$$\begin{aligned} V_1^a &= \\ & \left\{ \phi_a \in W^{1,2}(\Omega \times \Omega \times [0, t_f]; \mathbb{C}) : \nabla_x \phi_a \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \times \Omega \times [0, t_f], \right. \\ & \nabla_y \phi_a \cdot \mathbf{n} = 0 \text{ on } \Omega \times \partial\Omega \times [0, t_f], \\ & \left. \frac{\partial \phi_a(x, y, 0)}{\partial t} = 0 \text{ and } \frac{\partial \phi_a(x, y, t_f)}{\partial t} = 0 \text{ in } \Omega \times \Omega \right\} \end{aligned} \quad (8)$$

and

$$\begin{aligned} V_2^a &= \{ \phi_a^M \in W^{1,2}(\Omega \times [0, t_f]; \mathbb{C}) : \phi_a^M = 0 \text{ on } \partial(\Omega_1 \cup \Omega_2) \times [0, t_f], \\ & \phi_a^M(z, t) = \phi_{in}(z, t) \text{ on } \partial\Omega_{in} \times [0, t_f] \text{ and } \phi_a^M(z, 0) = \phi_0 \text{ in } \Omega \}, \end{aligned} \quad (9)$$

where

$$\partial\Omega = \partial\Omega_{in} \cup \partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_{out}.$$

Moreover we suppose such a union is quasi-disjoint and through $\partial\Omega_{in}$ and $\partial\Omega_{out}$ are allowed the entering and exiting of fluid mass, respectively.

We define also,

$$V_1^b = \left\{ \phi_b \in W^{1,2}(\Omega \times [0, t_f]; \mathbb{C}) : \nabla_x \phi_b \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \times [0, t_f], \right. \\ \left. \frac{\partial \phi_b(x, 0)}{\partial t} = 0 \text{ and } \frac{\partial \phi_b(x, t_f)}{\partial t} = 0 \text{ in } \Omega \right\} \quad (10)$$

and

$$V_2^b = \left\{ \phi_b^M \in W^{1,2}(\Omega \times [0, t_f]; \mathbb{C}) : \phi_b^M = 0 \text{ on } \partial(\Omega_1 \cup \Omega_2) \times [0, t_f], \right. \\ \left. \phi_b^M(z, t) = \phi_{in}(z, t) \text{ on } \partial\Omega_{in} \times [0, t_f] \text{ and } \phi_b^M(z, 0) = \phi_0 \text{ in } \Omega \right\}, \quad (11)$$

Observe that, we may denote

$$\rho(y, t) = \rho_a(y, t) + \rho_b(y, t).$$

2.1 The position fields

For the phase a , we start by denoting the macroscopic electronic position field as

$$\mathbf{r}_a : \Omega \times [0, t_f] \rightarrow \mathbb{R}^3,$$

where

$$\mathbf{r}_a(x, t) = (X_a(x, t), Y_a(x, t), Z_a(x, t)).$$

In a internal variable and relativistic context, the particle position field for such a phase a , is defined by

$$\hat{\mathbf{r}}_a(x, t) = (\mathbf{r}_a(x, t), t) + \tilde{\mathbf{r}}_a^p(\mathbf{r}_a(x, t), t) = (\mathbf{r}_a(x, t), t) + (ct, \tilde{\mathbf{r}}_a(\mathbf{r}_a(x, t))).$$

Here $c > 0$ denotes the speed of light.

For the phase b , we start by denoting the macroscopic electronic position field as

$$\mathbf{r}_b : \Omega \times [0, t_f] \rightarrow \mathbb{R}^3,$$

where

$$\mathbf{r}_b(x, t) = (X_b(x, t), Y_b(x, t), Z_b(x, t)).$$

In a internal variable and relativistic context, the particle position field for such a phase a , is defined by

$$\hat{\mathbf{r}}_b(x, t) = (\mathbf{r}_b(x, t), t) + \tilde{\mathbf{r}}_b^p(\mathbf{r}_b(x, t), t) = (\mathbf{r}_b(x, t), t) + (ct, \tilde{\mathbf{r}}_b(\mathbf{r}_b(x, t))).$$

We define also the following function space

$$\begin{aligned} V_3^a &= \{ \mathbf{r}_a \in W^{1,2}(\Omega \times [0, t_f]; \mathbb{R}^3) : \mathbf{r}_a = \mathbf{0} \text{ on } (\partial\Omega_1 \cap \partial\Omega_2) \times [0, t_f] \\ &\quad \text{and } \mathbf{r}_a(z, 0) = \mathbf{r}_0 \text{ in } \Omega \}, \end{aligned} \quad (12)$$

and assume

$$\tilde{\mathbf{r}}_a \in V_4 = W^{1,2}(\mathbb{R}^3; \mathbb{R}^3).$$

For the phase b , we define the following corresponding space

$$\begin{aligned} V_3^b &= \{ \mathbf{r}_b \in W^{1,2}(\Omega \times [0, t_f]; \mathbb{R}^3) : \mathbf{r}_b = \mathbf{0} \text{ on } (\partial\Omega_1 \cap \partial\Omega_2) \times [0, t_f] \\ &\quad \text{and } \mathbf{r}_b(z, 0) = \mathbf{r}_0 \text{ in } \Omega \}, \end{aligned} \quad (13)$$

and assume

$$\tilde{\mathbf{r}}_b \in V_4 = W^{1,2}(\mathbb{R}^3; \mathbb{R}^3).$$

2.2 About the system constraints

Denoting by $m_T = m_a(0) + m_b(0)$ the total system mass at $t = 0$, we have the following constraint

$$\begin{aligned} m_T &= m_a(0) + m_b(0) \\ &= m_a(t) + \int_0^t \int_{\partial\Omega} \rho_a(y, \tau) \mathbf{v}_a \cdot \mathbf{n} \, dS \, d\tau \\ &\quad + m_b(t) + \int_0^t \int_{\partial\Omega} \rho_b(y, \tau) \mathbf{v}_b \cdot \mathbf{n} \, dS \, d\tau \end{aligned} \quad (14)$$

where

$$\begin{aligned} \mathbf{v}_a &= \mathbf{v}_a(\mathbf{r}_a(x, t), t), \\ \mathbf{v}_b &= \mathbf{v}_b(\mathbf{r}_b(x, t), t). \end{aligned}$$

and where

$$m_a(t) = \int_{\Omega} \rho_a(y, t) \, dy$$

and

$$m_b(t) = \int_{\Omega} \rho_b(y, t) \, dy,$$

on $[0, t_f]$.

Here $\mathbf{n}$ denotes outward normal field to $\partial\Omega = S$.

Relating the internal kinetics energy, in a relativistic context, we define the following functional

$$\begin{aligned}
& E_{C_1}^a \\
&= -\frac{1}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} \frac{\hat{\rho}_a(x, y, t)}{\sqrt{1 - \frac{\tilde{v}_a^2}{c^2}}} \frac{d\tilde{\mathbf{r}}_a}{dt} \cdot \frac{d\tilde{\mathbf{r}}_a}{dt} \sqrt{-g^a} |\det X'_a| dx dy dt \\
&= \frac{1}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} \frac{\hat{\rho}_a(x, y, t)}{\sqrt{1 - \frac{\tilde{v}_a^2}{c^2}}} (c^2 - \tilde{v}_a^2) \sqrt{-g^a} |\det X'_a| dx dy dt \\
&= \frac{1}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} c \hat{\rho}_a(x, y, t) \sqrt{c^2 - \tilde{v}_a^2} \sqrt{-g^a} |\det X'_a| dx dy dt \\
&= \frac{1}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} c \hat{\rho}_a(x, y, t) \sqrt{c^2 - g_{jk}^a \frac{\partial(X_a)_j}{\partial t} \frac{\partial(X_a)_k}{\partial t}} \sqrt{-g^a} |\det X'_a| dx dy dt \\
&= \frac{1}{2} \int_0^{t_f} \int_{\Omega} e_C^a dy dt, \tag{15}
\end{aligned}$$

where generically, we have denoted, $X_0 = t$,

$$g_{jk}^a = \frac{\partial \tilde{\mathbf{r}}_a}{\partial (X_a)_j} \frac{\partial \tilde{\mathbf{r}}_a}{\partial (X_a)_k}$$

$g^a = \det\{g_{jk}^a\}$, where

$$w \cdot v = -w_0 v_0 + \sum_{k=1}^3 w_k v_k$$

for

$$w = (w_0, w_1, w_2, w_3)$$

and

$$v = (v_0, v_1, v_2, v_3) \in \mathbb{R}^4,$$

and

$$e_C^a = \int_{\Omega} c \hat{\rho}_a(x, y, t) \sqrt{c^2 - g_{jk}^a \frac{\partial(X_a)_j}{\partial t} \frac{\partial(X_a)_k}{\partial t}} \sqrt{-g^a} |\det X'_a| dx.$$

With analogous notation, the corresponding functional for the phase b is defined by

$$\begin{aligned}
& E_{C_1}^b \\
&= \frac{1}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} c \hat{\rho}_b(x, y, t) \sqrt{c^2 - g_{jk}^b \frac{\partial(X_b)_j}{\partial t} \frac{\partial(X_b)_k}{\partial t}} \sqrt{-g^b} |\det X'_b| dx dy dt \\
&= \frac{1}{2} \int_0^{t_f} \int_{\Omega} e_C^b dy dt, \tag{16}
\end{aligned}$$

where,

$$e_G^b = \int_{\Omega} c \hat{\rho}_b(x, y, t) \sqrt{c^2 - g_{jk}^b \frac{\partial(X_b)_j}{\partial t} \frac{\partial(X_b)_k}{\partial t}} \sqrt{-g^b} |\det X'_b| dx.$$

For the interacting and self-interacting energies, in a non-linear Schrödinger equation context, we define the following functional,

$$\begin{aligned} E_{in} = & \frac{\alpha_1^a}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} \rho_a(y, t)^2 dy dt \\ & - \frac{\beta_1^a}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} \rho_a(y, t) dy dt \\ & + \frac{\alpha_1^b}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} \rho_b(y, t)^2 dy dt \\ & - \frac{\beta_1^b}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} \rho_b(y, t) dy dt \end{aligned} \quad (17)$$

for appropriate $\alpha_1^a > 0$, α_1^b and $\beta_1^a, \beta_1^b > 0$.

On the other hand, we define

$$\begin{aligned} E_M = & \frac{1}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} \left(\frac{\gamma_1^a}{m_p^a} |\nabla \phi_a(x, y, t)|^2 \right) |\phi_a^M(y, t)|^2 dx dy dt \\ = & \frac{1}{2} \int_0^{t_f} \int_{\Omega} \int_{\Omega} \left(\frac{\gamma_1^b}{m_p^b} |\nabla \phi_b(x, t)|^2 \right) |\phi_b^M(y, t)|^2 dx dy dt, \end{aligned} \quad (18)$$

for appropriate constants $\gamma_1^a, \gamma_1^b > 0$.

Concerning the restrictions, already including the respective Lagrange multipliers, we define also the following functionals,

$$\begin{aligned} J_{Aux_1^a} = & \int_0^{t_f} \int_{\Omega} E_1^a(y, t) \left(\int_{\Omega} (|\phi_a(x, y, t)|^2 dx - m_p^a) \right) dy dt, \end{aligned} \quad (19)$$

$$\begin{aligned} J_{Aux_1^b} = & \int_0^{t_f} E_1^b(t) \left(\int_{\Omega} (|\phi_b(x, t)|^2 dx - m_p^b) \right) dt, \end{aligned} \quad (20)$$

$$\begin{aligned} J_{Aux_2} = & \int_0^{t_f} E_2(t) \left(m_T - \int_{\Omega} \rho_a(y, t) dy - \int_0^t \int_{\partial\Omega} \rho_a(y, \tau) \mathbf{v}_a \cdot \mathbf{n} dS d\tau \right. \\ & \left. - \int_{\Omega} \rho_b(y, t) dy - \int_0^t \int_{\partial\Omega} \rho_b(y, \tau) \mathbf{v}_b \cdot \mathbf{n} dS d\tau \right) dt. \end{aligned} \quad (21)$$

$$J_{Aux_3^a} = \int_0^{t_f} \int_{\Omega} E_3^a(x, t) \left(\mathbf{v}_a(\mathbf{r}_a(x, t), t) - \frac{\partial \mathbf{r}_a(x, t)}{\partial t} \right) dx dt,$$

$$J_{Aux_3^b} = \int_0^{t_f} \int_{\Omega} E_3^b(x, t) \left(\mathbf{v}_b(\mathbf{r}_b(x, t), t) - \frac{\partial \mathbf{r}_b(x, t)}{\partial t} \right) dx dt,$$

$$J_{Aux_4} = - \int_0^{t_f} \int_{\Omega} \hat{P} \left(\frac{d\rho}{dt} + \rho_a \operatorname{div} \mathbf{v}_a + \rho_b \operatorname{div} \mathbf{v}_b \right) dy dt, \quad (22)$$

Moreover, considering the pressure and temperature scalar fields, we define

$$\begin{aligned} P_a &= \frac{d(\hat{P}\rho_a)}{dt}, \\ P_b &= \frac{d(\hat{P}\rho_b)}{dt}, \end{aligned}$$

where

$$\hat{P} \in V_5$$

$$V_5 = \{ \hat{P} \in W^{1,2}(\Omega \times [0, t_f]) : P = P_a + P_b = P(\hat{P}, \rho) = P_0, \text{ on } \partial\Omega \times [0, t_f] \}.$$

and the temperature fields, T_a, T_b through the equations

$$(C_v)_a \rho_a T_a = e_C^a \quad (23)$$

so that

$$(C_v)_b \rho_b T_b = e_C^b \quad (24)$$

for an appropriate real constant $C_v > 0$.

At this point, we define the entropy differentials

$$dS_a = -\hat{K}_a \frac{\rho_a}{\rho} \ln \left(\frac{\rho_a}{e \rho} \right) dy dt,$$

$$dS_b = -\hat{K}_b \frac{\rho_b}{\rho} \ln \left(\frac{\rho_b}{e \rho} \right) dy dt,$$

and the functional

$$E^S = \int_0^{t_f} \int_{\Omega} (T_a dS_a + T_b dS_b).$$

Moreover, we also define

$$e^S = T_a S_a + T_b S_b,$$

where

$$S_a = -\hat{K}_a \frac{\rho_a}{\rho} \ln \left(\frac{\rho_a}{e \rho} \right)$$

and

$$S_b = -\hat{K}_b \frac{\rho_b}{\rho} \ln \left(\frac{\rho_b}{e \rho} \right).$$

With such results in mind, we define the restriction

$$\begin{aligned} & (C_v)_a \rho_a \frac{dT_a}{dt} + (C_v)_b \rho_b \frac{dT_b}{dt} \\ &= K_a \nabla^2 T_a + K_b \nabla^2 T_b \\ & \quad - P_a \operatorname{div} \mathbf{v}_a - P_b \operatorname{div} \mathbf{v}_b + \frac{de^S}{dt}, \end{aligned} \tag{25}$$

for an appropriate real constant $K_a > 0$, $K_b > 0$.

At this point, we define the following function space

$$\begin{aligned} V_6^a &= \{T_a \in W^{1,2}(\Omega \times [0, t_f]) : \\ & T_a(z, t) = T_1^a(z, t) \text{ on } \partial\Omega \times [0, t_f] \text{ and } T_a(z, 0) = T_0^a(z) \text{ in } \Omega\}, \end{aligned} \tag{26}$$

$$\begin{aligned} V_6^b &= \{T_b \in W^{1,2}(\Omega \times [0, t_f]) : \\ & T_b(z, t) = T_1^b(z, t) \text{ on } \partial\Omega \times [0, t_f] \text{ and } T_b(z, 0) = T_0^b(z) \text{ in } \Omega\}, \end{aligned} \tag{27}$$

and

$$\begin{aligned} V_7^a &= \{\mathbf{v}_a \in W^{1,2}(\mathbb{R}^3; \mathbb{R}^3) : \mathbf{v}_a = \mathbf{0} \text{ on } (\partial\Omega_1 \cup \partial\Omega_2) \times [0, t_f] \\ & \text{and } \mathbf{v}_a(z, 0) = \mathbf{v}_0^a \text{ on } \Omega\}. \end{aligned} \tag{28}$$

$$\begin{aligned} V_7^b &= \{\mathbf{v}_b \in W^{1,2}(\mathbb{R}^3; \mathbb{R}^3) : \mathbf{v}_b = \mathbf{0} \text{ on } (\partial\Omega_1 \cup \partial\Omega_2) \times [0, t_f] \\ & \text{and } \mathbf{v}_b(z, 0) = \mathbf{v}_0^b \text{ on } \Omega\}. \end{aligned} \tag{29}$$

With such restrictions and definitions in mind, we define also the following functionals,

$$J_{Aux_5^a} = \int_0^{t_f} \int_{\Omega} E_5^a(y, t) ((C_v)_a \rho_a T_a - e_C^a) dy dt \quad (30)$$

$$J_{Aux_5^b} = \int_0^{t_f} \int_{\Omega} E_5^b(y, t) ((C_v)_b \rho_b T_b - e_C^b) dy dt \quad (31)$$

and

$$\begin{aligned} J_{Aux_6} = & \int_0^{t_f} \int_{\Omega} E_6(y, t) \left((C_v)_a \rho_a \frac{dT_a}{dt} + (C_v)_b \rho_b \frac{dT_b}{dt} \right. \\ & - K_a \nabla^2 T_a - K_b \nabla^2 T_b \\ & \left. + P_a \operatorname{div} \mathbf{v}_a + P_b \operatorname{div} \mathbf{v}_b - \frac{de^S}{dt} \right) dy dt. \end{aligned} \quad (32)$$

Finally, we define the following functional,

$$\begin{aligned} J_{Aux_7^a} = & \int_0^{t_f} \int_{\Omega} E_{50}^a(y, t) \\ & \times \left(\rho_a(y, t) - \int_{\Omega} \left(\frac{|\phi_a(x, y, t)|^2}{m_p^a} \right) dx |\phi_a^M(y, t)|^2 \right) dy dt. \end{aligned} \quad (33)$$

$$\begin{aligned} J_{Aux_7^b} = & \int_0^{t_f} \int_{\Omega} E_{50}^b(y, t) \\ & \times \left(\rho_b(y, t) - \int_{\Omega} \left(\frac{|\phi_b(x, y, t)|^2}{m_p^b} \right) dx |\phi_b^M(y, t)|^2 \right) dy dt. \end{aligned} \quad (34)$$

3 The main functional

Here we define the functional J , by

$$\begin{aligned} J = & \frac{1}{2} \int_0^{t_f} \int_{\Omega} \rho_a \mathbf{v}_a \cdot \mathbf{v}_a dy dt - \int_0^{t_f} \int_{\Omega} \rho_a \mathbf{g} \cdot \mathbf{r}_a dy dt \\ & + \frac{1}{2} \int_0^{t_f} \int_{\Omega} \rho_b \mathbf{v}_b \cdot \mathbf{v}_b dy dt - \int_0^{t_f} \int_{\Omega} \rho_b \mathbf{g} \cdot \mathbf{r}_b dy dt. \end{aligned} \quad (35)$$

Denoting

$$V = V_1^a \times V_1^b \times V_2^a \times V_2^b \times V_3^a \times V_3^b \times V_5 \times V_6^a \times V_6^b \times V_7^a \times V_7^b,$$

the final variational formulation $J_{total} : V \rightarrow \mathbb{R}$ stands for

$$\begin{aligned}
 J_{total} = & -J + E_{C_1}^a + E_{C_1}^b + E_{in} \\
 & + E_M + J_{Aux_1^a} + J_{Aux_1^b} + J_{Aux_2} + J_{Aux_3^a} + J_{Aux_3^b} \\
 & J_{Aux_4} + J_{Aux_5^a} + J_{Aux_5^b} \\
 & + J_{Aux_6} + J_{Aux_7^a} + J_{Aux_7^b} - E^S.
 \end{aligned} \tag{36}$$

The objective of this section is complete.

4 Conclusion

In this article, we have presented a functional which the variation leads to a system of partial differential equations intended to model a Bose-Einstein condensation chemical reaction.

In a near future research we intend to address similar models in superfluidity.

References

- [1] J.F. Annet, Superconductivity, Superfluids and Condensates, 2nd edn. (Oxford Master Series in Condensed Matter Physics, Oxford University Press, Reprint, 2010)
- [2] L.D. Landau and E.M. Lifschits, A Course of Theoretical Physics, Vol. 5- Statistical Physics, part 1. (Butterworth-Heinemann, Elsevier, reprint 2008).
- [3] P. Constantin, C. Foias, Navier-Stokes equation, University of Chicago Press, Chicago, 1989.
- [4] C. De Lellis , E. Brué, D. Albritton, M. Colombo, V. Giri , M. Janisch and H. Kwon, Instability and Non-uniqueness for the 2D Euler Equations, after M. Vishik (AMS-219) Volume 219 in the series Annals of Mathematics Studies, Princeton University Press, New Jersey, 2024. <https://doi.org/10.1515/9780691257846>
- [5] M. Hamouda M, D. Han, C-Y. Jung , R. Temam, Boundary layers for the 3D primitive equations in a cube: the zero-mode, Journal of Applied Analysis and Computation. 2018; 8, No. 3, (873-889), DOI:10.11948/2018.873.

- [6] A. Giorgini, A. Miranville A, R. Temam, Uniqueness and regularity for the Navier-Stokes-Cahn-Hilliard system, SIAM J. of Mathematical Analysis (SIMA).2019; 51, 3, (2535-2574), <https://doi.org/10.1137/18M1223459>.
- [7] C. Foias, R.M. Rosa, R. Temam, Properties of stationary statistical solutions of the three-dimensional Navier-Stokes equations, J. of Dynamics and Differential Equations, Special issue in memory of George Sell. 2019; 31, 3, 1689-1741, <https://doi.org/10.1007/s10884-018-9719-2>.
- [8] R. Temam, Navier-Stokes Equations, AMS Chelsea, reprint (2001).
- [9] J.C. Tannehill, D.A. Anderson, R.H. Pletcher, *Computational Fluid Mechanics and Heat Transfer*, 2nd edn., Taylor and Francis, Philadelphia, PA, 1997.
- [10] J.C. Strikwerda, finite difference schemes and partial differential equations, second edition Philadelphia, SIAM, 2004.
- [11] F.S. Botelho, "Approximate numerical procedures for the Navier-Stokes system through the generalized method of lines" Nonlinear Engineering, vol. 13, no. 1, 2024, pp. 20220376. <https://doi.org/10.1515/nleng-2022-0376>
- [12] F. Botelho, Existence of solution for the Ginzburg-Landau system, a related optimal control problem and its computation by the generalized method of lines, Applied Mathematics and Computation, 2012; 218 (11976-11989).
- [13] F.S. Botelho, An approximate proximal numerical procedure concerning the generalized method of lines. Mathematics 2022, 10(16), 2950; <https://doi.org/10.3390/math10162950>.
- [14] F. Botelho Functional Analysis and Applied Optimization in Banach Spaces, (Springer Switzerland, 2014).
- [15] F.S. Botelho, Functional analysis, calculus of variations and numerical methods in physics and engineering. Boca Raton, Florida, USA CRC Taylor and Francis; 2020.
- [16] F.S. Botelho, Advanced Calculus and its Applications in Variational Quantum Mechanics and Relativity Theory, CRC Taylor and Francis, Florida, 2021.
- [17] S. Sheng and Z. Shao, Concentration of mass in the pressureless limit of the Euler equations of one-dimensional compressible fluid

flow, *Nonlinear Analysis, Real world applications*, 52, (2020), 103039.
<https://doi.org/10.1016/j.nonrwa.2019.103039>

- [18] R.A. Adams and J.F. Fournier, *Sobolev Spaces*, 2nd edn. (Elsevier, New York, 2003).

ON PARAMETERIZATION OF SOME PROBLEMS IN NUMBER THEORY

Sergey N. Artekha

Space Research Institute of RAS
117997 Moscow, Russia
sergey.artekha@gmail.com

Received January 12, 2026

Abstract

The article is devoted to the parametric solution of some Diophantine equations with two variables. In essence, we find an additional parametric relationship between the solution of the original equation and the simpler linear equation $ax-by = \pm 1$. The method of hidden parameters helps to obtain a parametric representation of integer solutions for a number of problems in number theory. The method consists in that, first, instead of each quantity included in the original equation, we introduce an additional number of parameters using a linear or non-linear function, which allows us to find both connections between the parameters and some parametric solution of the desired equation. Then we reset (nullify) the introduced linear or non-linear functional dependence (hidden parameters), which in the original formulation of the problem would mean just changing the letter designation of the parameters. However, as a result of these actions, a simplified parametric solution is obtained and the found relationship between the parameters remains. In this paper, the method is discussed in detail using the example of parametric solving such Diophantine equation as the Pell equation. The method of such parameterization is also applied to problems related to Pell's equation. The methods of parameterization are discussed for the Delaunay equation, the Nagell equation and similar equations of higher powers.

Keywords: Diophantine equations, Pell equation, Delaunay equation, Nagell equation, parametric solution

1 Introduction

In the article [1], a method of hidden parameters was proposed for solving the Pell equation in natural numbers. To significantly reduce the time of numerical calculations for specific numbers of A , a number of simplifications were carried out in this work for five different special cases, into which the general problem can be divided. Also, for these cases, algorithms for quickly filling out the solution table for the sequence of numbers A have been developed. In this paper, we are not interested in reducing the calculation algorithm for fixed particular cases. Let us recall the essence of the method for its further application in order to obtain parametric representations for the solutions of some problems of number theory. Instead of each quantity included in the original equation, we introduce an additional number of parameters using a linear or non-linear function, which allows us to find both relationships between the parameters and some parametric solution of the desired equation. After that, we can reset (nullify) the introduced additional linear or non-linear dependence (in fact, hidden parameters), which in the original formulation of the problem would mean just replacing the letter designation of the parameters. However, as a result of these actions, a simplified parametric solution is obtained and the found relationship between the parameters remains. But we can remain (without nullify) the introduced additional parameters, then the solution turns out to be more cumbersome, with a large number of parameters, but capturing a larger number of parametric solutions. As a result, we find a parametric relationship between the solution of the original equation and the simpler equation $ax - by = \pm 1$. We will demonstrate in detail a method for solving the Pell equation in a way different from that used in [1].

The main goal of this article is to develop the proposed method in the most general form, including applying it to higher-degree Diophantine equations [2-10], such as the Delaunay equation, the Nagell equation and similar equations, and to obtain parametric representations for their solutions.

2 Pell's equation and related topics

We consider another direct representation of the solution to Pell's equation

$$Ax^2 + 1 = y^2 \tag{1}$$

for the case

$$A = p_{01}p_{02}, \quad x = 2fg, \tag{2}$$

where all the variables introduced for factoring denote some coprime numbers (but they can turn out to be anything: prime, composite, and one). We rewrite (1) in the form of a system

$$\begin{aligned} y + 1 &= 2p_{01}f^2, \\ y - 1 &= 2p_{02}g^2, \end{aligned} \quad (3)$$

from which you can take any of the two expressions for the value of y and write the following condition:

$$p_{01}f^2 - p_{02}g^2 = 1. \quad (4)$$

If for fixed values of f and g (that is, for a fixed value of x) we find one solution to (4), then we will know all values of A (there are an infinite number of them), for which the same value of x will also be a solution:

$$A^i = p_1^i p_2^i, \quad p_1^i = p_{01} + i g^2, \quad p_2^i = p_{02} + i f^2, \quad i \in \mathbb{Z}. \quad (5)$$

We will make linear substitutions in equation (4):

$$p_{01} = ak + b, \quad p_{02} = ck + d, \quad f = rk + s, \quad g = tk + u. \quad (6)$$

We group the resulting equation according to degrees of k , and so that it is performed with any values of k , equate each expression with the appropriate degree of k to zero. To resolve the resulting system of three equations with senior degrees in integers, we make substitutions $t = zt_0$, $r = zr_0$ and get a solution:

$$a = 2zr_0t_0^3, \quad b = t_0^2(3ur_0 - st_0), \quad c = 2zr_0^3t_0, \quad d = r_0^2(3st_0 - ur_0), \quad (7)$$

and from the equation of the system with the zero degree of k , the condition follows:

$$ur_0 - st_0 = 1. \quad (8)$$

Thus, we received the connection of equation (1) for case (2) with the simplest linear equation (8). Note that the obtained solution is true for any k , including $k = 0$. Although in this particular case, our substitutions (6) would only mean simple changing the letter designation of variables (hidden parameters).

If, with fixed values of u and s , we find the solution r_0 and t_0 to the equation (8), then the solutions will also be

$$r_j = r_0 + js, \quad t_j = t_0 + ju, \quad j \in \mathbb{Z}. \quad (9)$$

Then the final relationship between the solution of the equation (8) and the solution (2) of the equation (1) is as follows:

$$x = 2((r_0 + js)k + s)((t_0 + ju)k + u), \quad (10)$$

$$\begin{aligned}
 A = & \left(2k(r_0 + js)(t_0 + ju)^3 + (t_0 + ju)^2(3u(r_0 + js) - s(t_0 + ju)) + \right. \\
 & \left. i((t_0 + ju)k + u)^2 \right) \times \\
 & \times \left(2k(r_0 + js)^3(t_0 + ju) + (r_0 + js)^2(3s(t_0 + ju) - u(r_0 + js)) + \right. \\
 & \left. i((r_0 + js)k + s)^2 \right). \quad (11)
 \end{aligned}$$

Here i, j, k are arbitrary integers. However, note that there are also some rational numbers that lead to natural values x, A . (This can be done, if we are interested in the question, for which values of A some fixed value of x is a solution.)

Thus, we obtain the following

Proposition 1 *All solutions to equation (1) for the case (2) are given by expressions (10), (11), where the parameters u and s are related by relation (8), and i, j, k are arbitrary integers.*

We can write another solution if, on the contrary, we fix r_0 and t_0 and find solutions u_0 and s_0 of the equation (8); the solutions also will:

$$u_j = u_0 + jt_0, \quad s_j = s_0 + jr_0, \quad j \in \mathbb{Z}. \quad (12)$$

As a result, another ultimate relationship between the solution of equation (8) and the solution (2) of equation (1) looks as follows:

$$x = 2(r_0k + s_0 + jr_0)(t_0k + u_0 + jt_0), \quad (13)$$

$$\begin{aligned}
 A = & (2kr_0t_0^3 + t_0^2(3(u_0 + jt_0)r_0 - (s_0 + jr_0)t_0) + i(t_0k + u_0 + jt_0)^2) \times \\
 & \times (2kr_0^3t_0 + r_0^2(3(s_0 + jr_0)t_0 - (u_0 + jt_0)r_0) + i(r_0k + s_0 + jr_0)^2). \quad (14)
 \end{aligned}$$

Here i, j, k are again arbitrary integers (but there are also some rational numbers leading to integers x, A).

Thus, we obtain the following

Proposition 2 *All solutions to equation (1) for the case (2) can be presented by expressions (13), (14), where the parameters u_0 and s_0 are related by relation (8), and i, j, k are arbitrary integers.*

As a solution to equation (8), you can choose any identity, for example:

$$n_1 \cdot n_1 n_2^2 - (n_1 n_2 - 1) \cdot (n_1 n_2 + 1) \equiv 1, \quad (15)$$

or more general identities, for example:

$$\begin{cases} n_1^k - (n_1 - 1)(n_1^{k-1} + \dots + 1) \equiv 1 \\ (n_2 + 1)(n_2^{2k} - \dots + 1) - n_2^{2k+1} \equiv 1. \end{cases} \quad (15^*)$$

We choose, which of the multipliers from the identity (15) (or from any similar identity, for example, (15*)) to assign for which of variables in (8). After that, we substitute them into the solutions (10), (11) or (13), (14). As a result, we obtain parametric solutions of Pell's equations. When substituting a set of specific numerical parameters into the resulting identic formula, some solutions may be repeated, and for a particular A , several different solutions may appear (of course, not all of these solutions turn out to be primitive - with a minimal non-zero natural x). Sequential filling of the solution table occurs quite fast.

A similar approach can be used to obtain parametric solutions not only for even solutions x from (2), but also for odd $x = fg$, which together covers all possible solutions. In this particular case, in equation (4), the number 2 will be on the right, and all solutions in (7) must be doubled. Then equation (8) turns out to be unchanged. All substitutions retain their former form, and the corresponding analogous solutions will be:

$$x = ((r_0 + js)k + s)((t_0 + ju)k + u), \quad (10^*)$$

$$A = [4k(r_0 + js)(t_0 + ju)^3 + 2(t_0 + ju)^2(3u(r_0 + js) - s(t_0 + ju)) + i((t_0 + ju)k + u)^2][4k(r_0 + js)^3(t_0 + ju) + 2(r_0 + js)^2(3s(t_0 + ju) - u(r_0 + js)) + i((r_0 + js)k + s)^2]; \quad (11^*)$$

$$x = (r_0k + s_0 + jr_0)(t_0k + u_0 + jt_0), \quad (13^*)$$

$$A = (4kr_0t_0^3 + 2t_0^2(3(u_0 + jt_0)r_0 - (s_0 + jr_0)t_0) + i(t_0k + u_0 + jt_0)^2) \times (4kr_0^3t_0 + 2r_0^2(3(s_0 + jr_0)t_0 - (u_0 + jt_0)r_0) + i(r_0k + s_0 + jr_0)^2). \quad (14^*)$$

The condition will coincide with equation (8) and all subsequent substitutions, for example, from the identity (15), (15*) or similar identities, remain fully valid.

Thus, we write the following

Proposition 3 *All solutions to equation (1) for the case of odd $x = fg$ can be presented by expressions (10*), (11*) and (13*), (14*), where the parameters u_0 , s_0 , r_0 and t_0 are related by relation $u_0r_0 - s_0t_0 = 1$, and i , j , k are arbitrary integers.*

Similarly, with the help of linear substitutions, one can search for a solution to the equation

$$ax^2 - by^2 + cxy = n$$

with a given right side and a fixed one of the coefficients on the left side.

Let us now turn to higher-degree equations. If the equation can be reduced to an equation of type (4) with linear coefficients, then its further solution is elementary. Let, for example, the same power n be present in the equation:

$$p_{01}f^n - p_{02}g^n = 1. \quad (16)$$

For example, the solution of the equation

$$Ax^n + 1 = y^2 \quad (17)$$

for the following two cases:

$$x = 2fg, \quad \begin{cases} 1) A = p_{01}^* p_{02}, p_{01} = 2^{n-2} p_{01}^* \\ 2) A = p_{01} p_{02}^*, p_{02} = 2^{n-2} p_{02}^* \end{cases} \quad (18)$$

reduces to such the equation (16), where all the variables introduced for factoring denote some coprime numbers. By analogy with quadratic equations, it is also easy to reduce the search for odd solutions x to the solution of the equation (16).

If for fixed values of f and g we find one (any) solution to (16), then we will know all solutions (all values of these quantities):

$$p_1^i = p_{01} + i g^n, \quad p_2^i = p_{02} + i f^n, \quad i \in \mathbb{Z}. \quad (19)$$

To find a solution, the following substitutions can be made in (16):

$$p_{01} = b + \sum_{j=1}^{n-1} a_j k^j, \quad p_{02} = d + \sum_{j=1}^{n-1} c_j k^j, \quad f = rk + s, \quad g = tk + u. \quad (20)$$

Next, you need to group the resulting equation by powers of k , and in order for it to be fulfilled for any values of k , equate each expression before the corresponding power of k to zero. After that, the system of linear equations is solved relative all coefficients in p_{01} and p_{02} .

However, in order to find one solution (after all, this is enough), you can do it easier. If we want to find a connection between the solutions of equation (16) and equation

$$uf - sg = 1, \quad (21)$$

then we raise the last equation to the power $2n-1$ and obtain the required quantities:

$$\begin{cases} p_{01} = u^{2n-1} f^{n-1} - C_{2n-1}^1 u^{2n-2} f^{n-2} s g + \dots + (-1)^{n-1} C_{2n-1}^{n-1} u^n s^{n-1} g^{n-1}, \\ p_{02} = -s^{2n-1} g^{n-1} + C_{2n-1}^1 u f s^{2n-2} g^{n-2} - \dots + (-1)^n C_{2n-1}^{n-1} u^{n-1} f^{n-1} s^n, \end{cases} \quad (22)$$

written in terms of the expansion coefficients of the Newton binomial, taking into account the symmetry of the coefficients. For the first case in (18), the number u must be even. For the second case in (18), the number s must be even. In any of these cases the number i in (19) must be $i = 2^{n-2} i_1$. The choice

of some identities, for example, of the type (15) or (15*) as solutions to the equation (21), allows us to write the solutions of the equations (16) in a parametric form. Obviously, we can use substitutions that do not change equation (21), for example: $u \rightarrow u + jg, s \rightarrow s + jf$, or $f \rightarrow f + ks, g \rightarrow g + ku$, where j, k are arbitrary integers.

Thus, we obtain the following

Proposition 4 *All solutions to equations (16) and (17) for the case (18) are given by expressions (19), (22), where the parameters u and s are related by relation (21), and i is an arbitrary integer.*

In essence, we have obtained all solutions to the following problem in integers: for what numbers A , a given even x is a solution to equation (17).

Similarly, you can act in the case of an equation with different powers:

$$p_{01}f^n - p_{02}g^k = 1. \quad (23)$$

For fixed values of f and g (knowing one solution), all solutions have the form:

$$p_1^i = p_{01} + ig^k, \quad p_2^i = p_{02} + if^n, \quad i \in \mathbb{Z}, \quad (24)$$

and the relationship with the solutions of equation (21) after raising to the power $n + k - 1$ is given by the expressions:

$$\begin{cases} p_{01} = u^{n+k-1}f^{k-1} - C_{n+k-1}^1 u^{n+k-2}f^{k-2}sg + \dots + (-1)^{k-1}C_{n+k-1}^{k-1}u^n s^{k-1}g^{k-1}, \\ p_{02} = (-s)^{n+k-1}g^{n-1} + (-s)^{n+k-2}C_{n+k-1}^1 uf g^{n-2} + \dots + (-s)^k C_{n+k-1}^k u^{n-1}f^{n-1}, \end{cases} \quad (25)$$

which are written in terms of the expansion coefficients of the Newton binomial. Again, choosing as solutions to the equation (21) some identities, for example, of the type (15) or (15*), we can write the solutions of the equations (23) in a parametric form.

Therefore, we get the following

Proposition 5 *All solutions to equation (23) are given by expressions (24), (25), where the parameters u and s are related by relation (21), and i is an arbitrary integer.*

Thus, if a pair of coefficients enters the original equation linearly (or the original equation can be reduced to a similar equation), then it is easy to find a connection between this equation and a linear equation of type (21) and obtain parametric solutions in symbolic form (in principle, cover all solutions). Note that in article [8], in fact, only some special cases of polynomial

parametrization of the Pell equation are presented. If there is only one such a coefficient or the coefficients enter in a non-linear way, then the problem may become more complicated.

3 The Delaunay equation

We write the Delaunay equation in integers:

$$x^3 + By^3 = 1. \quad (26)$$

We consider the case when y and B are not divisible by 3 and write down an arbitrary partition into coprime factors:

$$B = p_{01}p_{02}, \quad y = fg. \quad (27)$$

As a result, we have a system of equations where the left-hand sides of the equations are coprime (the case when they have a common divisor of 3 can be considered in a completely similar way):

$$\begin{cases} x - 1 = p_{01}f^3 \\ x^2 + x + 1 = p_{02}g^3. \end{cases} \quad (28)$$

Multiplying the second equation by 4, we rewrite it as:

$$(2x + 1)^2 + 3 \cdot 1^2 = 4p_{02}g^3. \quad (29)$$

Since there exists a single-valued division procedure for any form of the type

$$X^2 + 3 \cdot Y^2,$$

then all divisors on the right side of (29) must be represented in the same form:

$$p_{02} = a^2 + 3b^2, \quad g = q^2 + 3s^2. \quad (30)$$

Substituting these expressions and the expression x from the first equation of the system (28) into the equation (29), we obtain the following system of equations:

$$\begin{cases} \pm\{2p_{01}f^3 + 3\} = A_0q[q^2 - 9s^2] \mp 9B_0s[q^2 - s^2], \\ \pm 1 = B_0q[q^2 - 9s^2] \pm 3A_0s[q^2 - s^2], \end{cases} \quad (31)$$

where the signs on the right-hand sides of both equations of the system (31) are consistent with each other, the signs $\pm$ on the left-hand sides of both equations are completely arbitrary (each is independent of anything), and the notations with consistent signs are also introduced:

$$A_0 = a \mp 3b, \quad B_0 = a \pm b. \quad (32)$$

Let us include the sign $\mp$ from the right parts of the system (31) into the value s , then in the first equation of (31), the sign (+) will present on the right, and in the second equation of (31), the sign (-) will present on the right. Let us also include the sign $\pm$ on the left-hand side of the first equation of (31) in the definition of the value f . As a result, we rewrite the system (31) in the form:

$$\begin{cases} A_0 q[q^2 - 9s^2] + 9B_0 s[q^2 - s^2] - 2p_{01} f^3 = \pm 3, \\ B_0 q[q^2 - 9s^2] - 3A_0 s[q^2 - s^2] = \pm 1. \end{cases} \quad (31^*)$$

If one arbitrary solution A_0 and B_0 is known from the second equation of (31*) for the given values of q and s , then we can write down all such solutions:

$$A_i = A_0 + i(q^3 - 9qs^2), \quad B_i = B_0 + i(3q^2s - 3s^3), \quad i \in \mathbb{Z}, \quad (33)$$

and from (32) the values a and b can be expressed, i.e. all possible p_{02}^i values can be found.

We are looking for a solution to the second equation of the system (31*) in the form:

$$q = hk + q_0, \quad s = jk + s_0, \quad A_0 = gk^2 + rk + t, \quad B_0 = mk^2 + ck + d, \quad k \in \mathbb{N}. \quad (34)$$

Let us group the resulting equation by powers of k , and in order for it to be fulfilled for any values of k , we equate each expression before the corresponding power of k to zero. We solve the resulting system of six equations in integers with respect to the variables g, r, t, m, c, d and find the quantities A_0 and B_0 of interest to us. In order for them to be integers, we equate the literal expression in the denominator of these quantities to ± 1 and take it as some desired condition. In this case, to cancel 3 in the denominator of A_0 , it is necessary to put $h = 3h_0$. Any other choice would lead to a contradiction with the desired condition when compared modulo 3. Taking into account this substitution, we write the final desired condition as follows:

$$jq_0 - 3h_0s_0 = \pm 1. \quad (35)$$

To resolve this equation as a parametric identity, one can choose, similarly to (15) or (15*), any identity with three terms and, comparing with (35), choose from them the values for the variables j, q_0, h_0, s_0 . We also get the following result for the values of A_0 and B_0 :

$$\begin{aligned} A_0 = & \frac{81h_0^3k^2}{8}(9h_0^4 - 3h_0^2j^2 - 5j^4) + \\ & + \frac{27k}{8}(18h_0q_0 + 15h_0^4j^2q_0 - j^6q_0 - 63h_0^5js_0 - 30h_0^3j^3s_0 - 3h_0j^5s_0) + \\ & + \frac{9}{8}(9h_0^5q_0^2 + 10h_0^3j^2q_0^2 + 5h_0j^4q_0^2 - 15h_0^4jq_0s_0 - 30h_0^2j^3q_0s_0 - 3j^5q_0s_0 - \\ & - 72h_0^5s_0^2), \\ B_0 = & \frac{9jk^2}{8}(81h_0^6 + 45h_0^4j^2 + 3h_0^2j^4 - j^6) + \end{aligned} \quad (36)$$

$$\begin{aligned}
 & + \frac{9k}{8} (27h_0^5 j q_0 + 30h_0^3 j^3 q_0 + 7h_0 j^5 q_0 + 81h_0^6 s_0 - 15h_0^2 j^4 s_0 - 2j^6 s_0) + \\
 & + \frac{1}{8} (8j^5 q_0^2 + 243h_0^5 s_0 q_0 + 270h_0^3 j^2 s_0 q_0 + 15h_0 j^4 s_0 q_0 - 405h_0^4 j s_0^2 - \\
 & - 90h_0^2 j^3 s_0^2 - 9j^5 s_0^2). \quad (37)
 \end{aligned}$$

For all parameters, one can check all possible remainders modulo 8 and determine those linear sequences, with which expressions (36) and (37) become integers (the number 8 in the denominator is canceled). We will not write out all the sequences, since there are quite a lot of them. Note that in the method of hidden parameters, we can well restrict ourselves to the case $k = 0$, $q = q_0$, $s = s_0$, because it was enough for us to find at least one solution.

Let us substitute, instead of quantities A_0 and B_0 , an arbitrary solution (33) into the first equation of the system (31*):

$$i(q^2 + 3s^2)^3 - 2p_{01}f^3 = A_0 q[9s^2 - q^2] + 9B_0 s[s^2 - q^2] \pm 3. \quad (38)$$

In this equation, the right-hand side is fixed, since q and s are given, and the values A_0 and B_0 are found in (36) and (37). On the left-hand side of the equation (38), the values f , q , and s will be given. The general solution (38) can be written as follows:

$$\begin{cases} i^u = I_0(A_0 q[9s^2 - q^2] + 9B_0 s[s^2 - q^2] \pm 3) + 2uf^3, & u \in \mathbb{Z}, \\ p_{01}^u = \frac{P_0(A_0 q[9s^2 - q^2] + 9B_0 s[s^2 - q^2] \pm 3)}{2} + u(q^2 + 3s^2)^3, \end{cases} \quad (39)$$

where the quantities I_0 and P_0 (again, it is enough for us to find one solution) are found from the condition:

$$I_0(q^2 + 3s^2)^3 - P_0 f^3 = 1. \quad (40)$$

A solution of this equation

$$\begin{cases} P_0 = w^3(w^2 f^2 - 5wfQv + 10Q^2 v^2), \\ I_0 = v^3(Q^2 v^2 - 5Qvwf + 10w^2 f^2), \end{cases} \quad (41)$$

with the introduced notation

$$Q = q^2 + 3s^2, \quad (42)$$

can be written through the solution of a more simple equation:

$$v(q^2 + 3s^2) - wf = 1. \quad (43)$$

To present this equation as a parametric identity, we can again choose some identities (see, for example, (15) or (15*), etc.) and select from them values for the variables f , v , w .

Thus, we finally obtain the following technique for writing nontrivial parametric solutions for the Delaunay equation:

Algorithm 1

- (1) By specifying q_0 and s_0 , we find the values of j and h_0 from the equation (35), and
- (2) from equalities (36) and (37) we obtain the values of A_0 and B_0 .
- (3) With the help of substitutions (33), we find all the values of A_i and B_i , and
- (4) from the system (32) all possible $a_i = (3B_i + A_i)/4$ and $b_i = \pm(B_i - A_i)/4$ follow, therefore,
- (5) we have obtained all possible values of p_{02}^i and g^k from (30).
- (6) Further, by setting the value of f , we find the values of v and w from the equation (43).
- (7) The first expression in (41) gives the value of P_0 , and
- (8) from the second expression (39) we find all possible values of p_{01}^u .
- (9) As the result, for a given value of y , we found all possible values of B from (27). The corresponding x values are found from the first equation (28).

Note that knowing a solution of the equation (35) allows us to find another, infinite sequence of non-trivial solutions:

$$\begin{cases} q_n = q_0 + 3nh_0, \\ s_n = s_0 + nj, \quad n \in \mathbb{Z}, \end{cases} \quad (44)$$

and then we proceed according to the method described in the previous paragraph.

In both approaches, filling the table of nontrivial solutions for different values of B occurs elementarily. Note that for the parametric representation of solutions to the Delaunay equation, it is easy to search for solutions (35) in the form of identities, but it may be difficult to simultaneously find identical representations for solutions to equation (43).

4 Nagell's equation

We write the Nagell equation in integers:

$$x^4 - By^4 = \pm 1. \quad (45)$$

Here we consider the case with the upper (+) sign and write down an arbitrary partition into coprime factors:

$$B = P_{01}P_{02}, \quad y = fg. \quad (46)$$

For the case of all odd factors, as a result, we have a system of equations:

$$\begin{cases} x^2 - 1 = P_{01}f^4, \\ x^2 + 1 = P_{02}g^4. \end{cases} \quad (47)$$

Since there exists a single-valued division procedure for a shape of the form

$$X^2 + Y^2,$$

then all divisors on the right-hand side of the second equation of the system (47) should be represented in such the form:

$$P_{02} = a^2 + b^2, \quad g = q^2 + s^2. \quad (48)$$

Substituting these expressions into the second equation of the system (47), we obtain the following solution x and the condition:

$$\begin{cases} x = 4aqs(q^2 - s^2) \pm b[(q^2 - s^2)^2 - (2qs)^2], \\ a[(q^2 - s^2)^2 - (2qs)^2] \mp 4bqs(q^2 - s^2) = \pm 1. \end{cases} \quad (49)$$

Here, the signs on the left-hand side of the condition are consistent with the signs in the solution, and the signs on the right-hand side of the condition are completely arbitrary (each does not depend on anything).

For the given values of q and s , if one arbitrary solution (a_0, b_0) is known from the second equation (49), then we can write down all such solutions:

$$a_i = a_0 \pm 4iqs(q^2 - s^2), \quad b_i = b_0 + i[(q^2 - s^2)^2 - (2qs)^2], \quad i \in Z, \quad (50)$$

and from the first equation in (49) we obtain all possible x_i , and from (48) we can find all possible values of P_{02}^i . The second equation in (49) can be solved by the method of hidden parameters: by setting the values q and s as linear dependencies in k and looking for the values a and b in the form of quadratic dependencies; and then we separately equate the resulting expressions before different powers of k to zero (after finding the solution, you can choose $k = 0$). The result is the following solution:

$$a_0 = \frac{1}{4}qrs^2(7p^6 + 35p^4r^2 + 21p^2r^4 + 5r^6) -$$

$$-\frac{1}{4}pq^2s(5p^6 + 21p^4r^2 + 35p^2r^4 + 7r^6) + p^7s^3 - q^3r^7, \quad (51)$$

$$\begin{aligned} b_0 = \frac{\mp 1}{16} [pq^3(5p^6 + 21p^4r^2 + 35p^2r^4 + 35r^6) - q^2rs \times \\ (7p^6 + 35p^4r^2 + 105p^2r^4 + 29r^6) - pqs^2 \times \\ (29p^6 + 105p^4r^2 + 35p^2r^4 + 7r^6) +] \\ + rs^3(35p^6 + 35p^4r^2 + 21p^2r^4 + 5r^6). \end{aligned} \quad (52)$$

In the last expression, the $\mp$ signs are consistent with the first expression in (49) and expression (50). The values of p and r are found from the following condition:

$$ps - qr = \pm 1, \quad (53)$$

in which the signs on the right-hand side are completely arbitrary. Again, all variables from this equation can be selected by comparison with parametric identities of the type (15), (15*), or others.

We introduce the following notations:

$$h = 4qs(q^2 - s^2), d = (q^2 - s^2)^2 - (2qs)^2, m = h^2 \pm d^2, N = a_0h \pm b_0d. \quad (54)$$

As a result, we write the solution in the form:

$$x_i = N + im, \quad P_{02}^i = a_i^2 + b_i^2. \quad (55)$$

However, we still need to coordinate this solution with the first equation of the system (47), from which two equations can be obtained:

$$\begin{cases} x - 1 = P_{11}f_1^4, & f = f_1f_2, \\ x + 1 = P_{12}f_2^4, & P_{01} = P_{11}P_{12}. \end{cases} \quad (56)$$

Subtracting the first equation from the second equation of (56), we obtain:

$$P_{12}f_2^4 - P_{11}f_1^4 = 2. \quad (57)$$

The solution to this equation in the general form will be:

$$\begin{cases} P_{12}^j = 2P_{12}^0 + jf_1^4, \\ P_{11}^j = 2P_{11}^0 + jf_2^4, \end{cases} \quad (58)$$

where $j \in \mathbb{Z}$ and

$$\begin{cases} P_{12}^0 = 35c^4f_1^3r^3 - 21c^5f_1^2f_2r^2 + 7c^6f_1f_2^2r - c^7f_2^3, \\ P_{11}^0 = 35c^3f_2^3r^4 - 21c^2f_1f_2^2r^5 + 7cf_1^2f_2r^6 - r^7f_1^3, \end{cases} \quad (59)$$

and the values of c and r are found from the condition:

$$rf_1 - cf_2 = 1. \quad (60)$$

These variables can again be taken from a comparison with an arbitrary parametric identity of the type (15), (15*). Next, it follows from the first equation of (56) that

$$x_j = 1 + 2P_{11}^0f_1^4 + jf_1^4f_2^4. \quad (61)$$

Now we need to coordinate the solutions (61) and (55), i.e. find such values i and j , for which the same solutions x are obtained. Introducing the notations

$$E = 1 + 2P_{11}^0f_1^4 - N, \quad F = f_1^4f_2^4 \quad (62)$$

and equating (61) and (55), we obtain the equation

$$im - jF = E, \quad (63)$$

the solution of which in the general form (for $E \neq 0$) will be the system:

$$\begin{cases} i = Ei_0 + nF, \\ j = Ej_0 + nM, \end{cases} \quad (64)$$

$n \in \mathbb{Z}$, and values of i_0 and j_0 can be found from the condition:

$$i_0 m - j_0 F = 1. \quad (65)$$

If it turns out that $E = 0$, then in this particular case, it will be $i = F, j = m$.

Thus, for writing parametric chains of solutions for the Nagell equation, we finally formulate

Algorithm 2

- I. We give arbitrary values of q and s and find corresponding values of p and r from the equation (53) (even identical substitutions can be used at this stage).
- II. After that, from the expressions (51) and (52), we obtain the values of a_0 and b_0 , and from the definitions (54) we calculate the values of h, d, m и N .
- III. By setting arbitrary values of f_1 and f_2 , we find the quantities r and c from the equation (60), and from the expressions (59) we find the quantities P_{11}^0 and P_{12}^0 .
- IV. Using the definitions (62) and solving the equation (65), we find values of i_0 and j_0 .
- V. From the expression (64), we find values of i and j , and from the expression (55) we obtain values of x_i and P_{02}^i .
- VI. From the expression (58), we find values of P_{11}^j и P_{12}^j , therefore, we know P_{01}^j .
- VII. Using the remaining expressions from (56), (48) and (46), we find B_i and y_i .

We will not consider cases where any factors in (46) are even - they can easily be resolved in a completely similar way (reduced to dividing some P and a number of expressions obtained by some powers of 2).

If in the equation (45) with the minus sign on the right-hand side, instead of the fourth power of x , there would a second power: $x^2 - By^4 = -1$, then such an equation was already resolved using expressions (49) - (53). Moreover, the proposed technique allows one to analogously solve a similar equation with an arbitrary degree of y : $x^2 - By^n = -1$. But, unfortunately, the case with the minus sign on the right-hand side of the original equation (45) leads to the need to look for squares in some linear sequence or look for the possibility of its integer division by the fourth power of a certain number, which greatly complicates the search for a general parametric representation of the solution.

A similar difficulty with finding the possibility for a linear sequence to be divided by the fifth power of a certain number has the solving of an equation of the fifth degree in x and y , analogous to the Delaunay and Nagell equations: $x^5 - By^5 = -1$.

5 Conclusion

Thus, a method has been proposed that makes it possible to obtain various parametric representations of solutions to Diophantine equations. Using factorization and/or quadratic representations, we break the original equation into equations that have a pair of unknown free variables. And then we look for a relation between the new equations and a simpler linear equation of the type $ax - by = \pm 1$. This can be achieved either by raising the last equation to a certain power and equating the required variables with the corresponding expressions, or by using the method of hidden parameters. This method is as follows. Instead of each quantity included in the equation under study, we introduce an additional number of parameters using a linear or non-linear function, which allows us to find connections between the parameters, as well as some parametric solution of the desired equation. After that, we can reset (nullify) the introduced additional linear or non-linear dependence (in fact, hidden parameters), which in the original formulation of the problem would mean just replacing the letter designation of the parameters. However, as a result of these actions, a simplified parametric solution is obtained and the found relationship between the parameters remains. But if we do not reset (not nullify) the introduced additional parameters, then the solution turns out to be more cumbersome, with a large number of parameters, but including a larger number of parametric solutions. In any case, the solution is found through the solution of a linear equation of the type $ax - by = \pm 1$.

The proposed solution method was demonstrated in detail when solving the Pell equation, and also developed in a more general form and applied to the parametric solution of high-order Diophantine equations, such as the Delaunay equation, the Nagell equation and similar equations.

References

1. Arteha S.N.: Method of hidden parameters and Pell's equation. JP Journal of Algebra, Number Theory and Applications 2(1), 21-46 (2002)
<http://doi.org/10.5281/zenodo.3463927>

2. Andreescu T., Andrica D.: Quadratic Diophantine Equations. Springer, New York (2015) <https://doi.org/10.1007/978-0-387-54109-9>
3. Delaunay B.N., Faddeev D.K.: The theory of irrationalities of the third degree. Amer. Math. Soc. Providence, Rhode Island (1964) <https://www.mathnet.ru/eng/tm/v11/p3>
4. Jia Ch., Matsumoto K.: Analytic Number Theory. Springer, New York (2002) <https://doi.org/10.1007/978-1-4757-3621-2>
5. Matthews K.R., Robertson J.P., Srinivasan A.: On fundamental solutions of binary quadratic form equations. Acta Arithmetica **169**(3), 291-299 (2015) <http://doi.org/10.4064/aa169-3-4>
6. Nagell T.: Contributions to the theory of a category of Diophantine equations of the second degree with two unknowns. Nova Acta Regiae Soc. Sci. Upsaliensis **16**(2), Ser. 4, 38 (1955)
7. Nagell T.: Introduction to Number Theory. AMS Chelsea Publishing **163**, New York (1964) <https://archive.org/details/introductiontonu0000nagel/page/n5/mode/2up>
8. Scherr Z., Thompson K.: Quartic integral polynomial Pell equations. Journal of Number Theory **259**, 38–56 (2024) <https://doi.org/10.1016/j.jnt.2023.12.009>
9. Togbé A., Walsh P.G.: A classical approach to a parametric family of simultaneous Pell equations with applications to a family of Thue equations. Bol. Soc. Mat. Mex. **28**, 4 (2022) <https://doi.org/10.1007/s40590-021-00393-5>
10. Tzanakis N., de Weger B.M.M.: On the practical solution of the Thue equation. J. Number Theor **31**(2), 99-132 (1989) [https://doi.org/10.1016/0022-314X\(89\)90014-0](https://doi.org/10.1016/0022-314X(89)90014-0)

**STEADY-STATE ENTANGLEMENT CLASS IN
FINITE-DIMENSIONAL QUANTUM SYSTEMS CANNOT
BE DETERMINED BY FULL NON-ZERO LIOUVILLIAN
SPECTRA**

Christian G. Barker
Independent Scholar
Cornwall, United Kingdom
smartion1971@proton.me

Received February 19, 2026

Abstract

We construct two finite-dimensional, exactly solvable two-qubit Lindbladian systems possessing strict two-dimensional invariant manifolds. For both models, we derive closed-form steady states, complete sets of non-zero Liouvillian eigenvalues with multiplicities, and analytic concurrence expressions. We show that, under parameter matching, the two generators possess identical full non-zero Liouvillian spectra, including both relaxation rates (real parts) and coherent oscillation frequencies (imaginary parts), while converging to steady states with locally unitarily inequivalent entanglement structure. This establishes constructively that full non-zero Liouvillian spectral data are insufficient to determine steady-state entanglement class, because the non-zero spectrum governs contraction transverse to the invariant manifold rather than its internal geometric resource structure.

AMS Classification 81P45, 81S22, 37N20

Keywords. Liouvillian spectra, steady-state entanglement, Lindbladian dynamics, invariant manifolds

1 Introduction

Quantum systems with environmental coupling evolve according to completely positive trace-preserving semigroups generated by Lindblad operators. The spectrum of the Liouvillian generator governs transient dynamics toward steady states: real parts of eigenvalues set relaxation rates, with the smallest non-zero real part defining the Liouvillian gap, while imaginary parts encode coherent oscillatory structure.

Spectral properties of the Liouvillian are widely used as dynamical diagnostics in dissipative physics, including relaxation-time analysis, dissipative criticality, and resource persistence. It is therefore natural to ask whether the full non-zero Liouvillian spectrum, encompassing both relaxation and oscillatory timescales, encodes sufficient information to determine the geometric and entanglement structure of the steady state.

We provide a constructive negative answer. Two analytically solvable two-qubit Lindbladians are presented whose full non-zero eigenvalue sets coincide exactly under parameter matching, yet whose steady states possess inequivalent entanglement properties.

2 Structural Framework

Let a finite-dimensional Lindbladian admit a decomposition of operator space

$$\mathcal{B}(\mathcal{H}) = \mathcal{M} \oplus \mathcal{M}^\perp, \quad (1)$$

such that:

1. The invariant manifold $\mathcal{M}$ contains the unique steady state.
2. Dissipators produce strict contraction on $\mathcal{M}^\perp$.
3. The Hamiltonian acts only within $\mathcal{M}$.

In Liouville representation, the generator admits block form

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_\mathcal{M} & 0 \\ * & \mathcal{L}_\perp \end{pmatrix}. \quad (2)$$

The non-zero eigenvalues are determined entirely by contraction on transverse sectors, while steady-state geometric resources are fixed by the kernel structure on the invariant manifold.

3 Model I: Bell-Invariant Manifold

Bell basis:

$$|\psi^\pm\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}, \quad (3)$$

$$|\phi^\pm\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}. \quad (4)$$

Invariant manifold:

$$\mathcal{M}_1 = \text{span}\{|\psi^-\rangle, |\phi^-\rangle\}. \quad (5)$$

Dissipators:

$$L_1 = \sqrt{\gamma} |\psi^-\rangle \langle \psi^+|, \quad (1)$$

$$L_2 = \sqrt{\gamma} |\psi^-\rangle \langle 00|, \quad (2)$$

$$L_3 = \sqrt{\gamma} |\psi^-\rangle \langle 11|. \quad (3)$$

Hamiltonian:

$$H_1 = \epsilon(\sigma_x^{(1)} - \sigma_x^{(2)}). \quad (6)$$

Assume $\gamma > 0$ and, without loss of generality, $\epsilon \geq 0$ (a sign flip of ϵ changes only an internal phase convention). Define $r = \epsilon/\gamma \geq 0$. The steady state in $\mathcal{M}_1$ is

$$\rho_1 = \begin{pmatrix} a(r) & iy(r) \\ -iy(r) & 1 - a(r) \end{pmatrix}, \quad (7)$$

where

$$a(r) = \frac{16r^2 + 1}{32r^2 + 1}, \quad y(r) = -\frac{4r}{32r^2 + 1}. \quad (8)$$

Concurrence:

$$C_1(r) = \frac{\sqrt{1 + 64r^2}}{32r^2 + 1}. \quad (9)$$

Full non-zero Liouvillian spectrum (with multiplicities):

$$\lambda_a = -\gamma \quad (\text{mult. } 4), \quad (4)$$

$$\lambda_b = -\frac{\gamma}{2} \quad (\text{mult. } 1), \quad (5)$$

$$\lambda_{c,\pm} = -\frac{3\gamma}{4} \pm \frac{\gamma}{4} \sqrt{1 - 64r^2} \quad (\text{each mult. } 4), \quad (6)$$

$$\lambda_{d,\pm} = -\frac{3\gamma}{4} \pm \frac{\gamma}{4} \sqrt{1 - 256r^2} \quad (\text{each mult. } 1), \quad (7)$$

plus a single zero eigenvalue.

For $r > 1/8$, the eigenvalues $\lambda_{c,\pm}$ acquire non-zero imaginary parts and form complex conjugate pairs. Similarly, for $r > 1/16$, the eigenvalues $\lambda_{d,\pm}$ become complex conjugate pairs.

4 Model II: Spectrum-Matched Computational Manifold

Define a fixed nonlocal unitary U by

$$U|\psi^-\rangle = |00\rangle, \quad (8)$$

$$U|\phi^-\rangle = |11\rangle, \quad (9)$$

$$U|\psi^+\rangle = |01\rangle, \quad (10)$$

$$U|\phi^+\rangle = |10\rangle. \quad (11)$$

Construct Model II by conjugation:

$$H_2 = UH_1U^\dagger, \quad L_k^{(2)} = UL_kU^\dagger. \quad (10)$$

An equivalent explicit representation is:

$$H_2 = -2\epsilon(|00\rangle\langle 11| + |11\rangle\langle 00|), \quad (12)$$

$$J_1 = \sqrt{\gamma}|00\rangle\langle 01|, \quad (13)$$

$$J_2 = \sqrt{\gamma}|00\rangle\langle 10|, \quad (14)$$

$$J_3 = \sqrt{\gamma}|00\rangle\langle 11|. \quad (15)$$

Because the superoperators satisfy

$$\mathcal{L}_2 = \mathcal{U}\mathcal{L}_1\mathcal{U}^{-1}, \quad (11)$$

Models I and II are exactly isospectral on full Liouville space, including all non-zero eigenvalues and multiplicities.

The steady state is $\rho_2 = U\rho_1U^\dagger$, with concurrence

$$C_2(r) = \frac{8r}{32r^2 + 1}. \quad (12)$$

5 Spectral Equivalence and Entanglement Inequivalence

Proposition 5.1. *For every fixed $r \geq 0$, Models I and II have identical full non-zero Liouvillian spectra (including multiplicities, real parts, and imaginary parts), but their steady states are not locally unitarily equivalent. In particular,*

$$C_1(r) - C_2(r) = \frac{\sqrt{1 + 64r^2} - 8r}{32r^2 + 1} > 0. \quad (13)$$

Proof. Similarity $\mathcal{L}_2 = \mathcal{U}\mathcal{L}_1\mathcal{U}^{-1}$ implies equality of full spectra, hence equality of all non-zero spectral data and multiplicities. On the other hand, concurrence is invariant under local unitaries on two qubits. Since $r \geq 0$ we have $\sqrt{1 + 64r^2} > \sqrt{64r^2} = 8r$, so the numerator is strictly positive and the denominator is positive. Therefore $C_1(r) > C_2(r)$ for all $r \geq 0$, proving local-unitary inequivalence of the steady states despite complete non-zero spectral coincidence.

At $r = 1/4$ this gives

$$C_1(1/4) = \frac{\sqrt{5}}{3}, \quad C_2(1/4) = \frac{2}{3}, \quad (14)$$

consistent with the strict inequality above. $\square$

6 Conclusion

We have provided an explicit finite-dimensional counterexample showing that full non-zero Liouvillian spectra, including real and imaginary parts with multiplicities, are insufficient to classify steady-state entanglement class. The non-zero spectrum encodes transverse contraction structure; steady-state entanglement is a kernel-manifold geometric property.

References

- [1] G. LINDBLAD, *Commun. Math. Phys.* **48** (1976) 119–130.
- [2] V. GORINI, A. KOSSAKOWSKI, AND E. C. G. SUDARSHAN, *J. Math. Phys.* **17** (1976) 821–825.

- [3] H.-P. BREUER AND F. PETRUCCIONE, *Oxford University Press*, Oxford (2002).
- [4] T. PROSEN, *New J. Phys.* **10** (2008) 043026.
- [5] F. VERSTRAETE, M. M. WOLF, AND J. I. CIRAC, *Nat. Phys.* **5** (2009) 633–636.
- [6] W. K. WOOTTERS, *Phys. Rev. Lett.* **80** (1998) 2245–2248.

MODIFIED QUATERNION AND OCTONION
DIVISION ALGEBRAS

S. C. Althoen
Dexter, Michigan
althoen@ymail.com

Received January 23, 2026

Abstract

We consider two classes of real division algebras that arise from modifications of the quaternions and the octonions. Since a *division algebra* is defined to be an algebra in which the equations $ux = v$ and $yu = v$ have unique solutions for all vectors v and nonzero vectors u , it follows that an algebra is a division algebra if and only if left translation by any nonzero vector u is invertible. That can be verified by checking that its determinant is positive definite. Our approach is elementary and derives from routine considerations of the algebras' multiplication tables. The quaternions and octonions are obtained from my tables by setting all $a_i = 1$ and all $b_{ij} = 0$.

Modified Quaternion Algebras

Let $\{1, e_1, e_2, e_3\}$ be a basis for the modified quaternions defined by the multiplication Table (1).

I have suppressed the identity row and column and the basis element, 1. For example, $b_{12} + e_3$ is shorthand for $b_{12}1 + e_3$ and $a + be_1 + ce_2 + de_3$ is shorthand for $a1 + be_1 + ce_2 + de_3$.

	1	e_1	e_2	e_3
(1)	e_1	$-a_1$	$b_{12} + e_3$	$b_{13} - e_2$
	e_2	$-b_{12} - e_3$	$-a_2$	$b_{23} + e_1$
	e_3	$-b_{13} + e_2$	$-b_{23} - e_1$	$-a_3$

Let L denote left translation by the vector $u = a + be_1 + ce_2 + de_3$.

Then $\det(L) = (a^2 + b^2 + c^2 + d^2)[a^2 + a_1b^2 + a_2c^2 + a_3d^2]$ independent of b_{12}, b_{13}, b_{23} . We have

Theorem 1. The modified quaternion algebra in Table (1) is a division algebra if and only if $a_1, a_2, a_3 > 0$.

Definition. An algebra A is *alternative* if for all x, y in A , $x(yx) = (xy)x$ and $(xx)y = x(xy)$.

Theorem 2. The algebra with Table (1) is alternative if and only if it is the quaternions.

Proof. We have $e_i(e_j e_j) = -a_j e_i$ and for some k , $(e_i e_j) e_j = (b_{ij} \pm e_k) e_j = b_{ij} \pm b_{ij} e_j - e_i$. It follows that the algebra with Table (1) is alternative if and only if $b_{12} = b_{13} = b_{23} = 0$ and $a_1 = a_2 = a_3 = 1$.

Modified Octonion Algebras

In the introduction of his “octavic system” in 1881 [1], Cayley considers the 8-dimensional real vector space $\mathbb{R}^8$ with basis vectors symbolized in a confusing way no one would choose today. His basis consists of vectors denoted by the symbols $0, 1, 2, 3, 4, 5, 6, 7$ where 0 denotes what he refers to as the “positive unity,” *i.e.*, what we denote by 1. He refers to the vectors $1, 2, 3, 4, 5, 6, 7$ as “the seven imaginaries.”

This vector space becomes an algebra under a multiplication Cayley adapts from the quaternions. To begin, $0^2 = 0$ and $1^2 = 2^2 = 3^2 = 4^2 = 5^2 = 6^2 = 7^2 = -0$. Recall that Cayley uses the symbol 0 to represent 1 and the symbols $1, \dots, 7$ to represent basis vectors. He then sets

$$\varepsilon_i = \pm 1 \text{ for } i = 1, \dots, 7.$$

To define the product of distinct pairs of vectors, he uses a compressed notation that in the quaternions would be written “ $ijk = \varepsilon$.” That compact equation summarizes the six equations

$jk = \varepsilon i, ki = \varepsilon j, ij = \varepsilon k, kj = -\varepsilon i, ik = -\varepsilon j$, and $ji = -\varepsilon k$, where, for the quaternions $\varepsilon = +1$. This representation corresponds to the familiar circle mnemonic for quaternion multiplication:



The octonion multiplication table is completed by the seven compact equations:

$$123 = \varepsilon_1, 145 = \varepsilon_2, 167 = \varepsilon_3, 246 = \varepsilon_4, 257 = \varepsilon_5, 347 = \varepsilon_6, 356 = \varepsilon_7.$$

We next present multiplication in tabular form:

	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	-0	$\varepsilon_1 3$	$-\varepsilon_1 2$	$\varepsilon_2 5$	$-\varepsilon_2 4$	$\varepsilon_3 7$	$-\varepsilon_3 6$
2	2	$-\varepsilon_1 3$	-0	$\varepsilon_1 1$	$\varepsilon_4 6$	$\varepsilon_5 7$	$-\varepsilon_4 4$	$-\varepsilon_5 5$
3	3	$\varepsilon_1 2$	$-\varepsilon_1 1$	-0	$\varepsilon_6 7$	$\varepsilon_7 6$	$-\varepsilon_7 5$	$-\varepsilon_6 4$
4	4	$-\varepsilon_2 5$	$-\varepsilon_4 6$	$-\varepsilon_6 7$	-0	$\varepsilon_2 1$	$\varepsilon_2 2$	$\varepsilon_6 3$
5	5	$\varepsilon_2 4$	$-\varepsilon_5 7$	$-\varepsilon_7 6$	$-\varepsilon_2 1$	-0	$\varepsilon_7 3$	$\varepsilon_5 2$
6	6	$-\varepsilon_3 7$	$\varepsilon_4 4$	$\varepsilon_7 5$	$-\varepsilon_4 2$	$-\varepsilon_7 3$	-0	$\varepsilon_3 1$
7	7	$\varepsilon_3 6$	$\varepsilon_5 5$	$\varepsilon_6 4$	$-\varepsilon_6 3$	$-\varepsilon_5 2$	$-\varepsilon_3 1$	-0

Cayley's interest was in solving the "eight square" problem [2]. To do this he determined conditions on the ε 's so that a sum of eight squares will be the product of two sums of eight squares. He first determines that without loss of generality one can take $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$ and then it follows that $-\varepsilon_4 = \varepsilon_5 = \varepsilon_6 = \varepsilon_7 = \pm 1$.

Either of these two choices of the ε_i guarantee that the algebra is a division algebra. See [2]. For specificity, I select $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1$ and $\varepsilon_5 = \varepsilon_6 = \varepsilon_7 = -1$ and then present a modification of Cayley's multiplication table as follows, suppressing the identity row and column and the basis vector, 1.

	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
	e_1	$-a_1$	$b_{12} + e_3$	$b_{13} - e_2$	$b_{14} + e_5$	$b_{15} - e_4$	$b_{16} + e_7$	$b_{17} - e_6$
	e_2	$-b_{12} - e_3$	$-a_2$	$b_{23} + e_1$	$b_{24} + e_6$	$b_{25} - e_7$	$b_{26} - e_4$	$b_{27} + e_5$
	e_3	$-b_{13} + e_2$	$-b_{23} - e_1$	$-a_3$	$b_{34} - e_7$	$b_{35} - e_6$	$b_{36} + e_5$	$b_{37} + e_4$
(2)	e_4	$-b_{14} - e_5$	$-b_{24} - e_6$	$-b_{34} + e_7$	$-a_4$	$b_{45} + e_1$	$b_{46} + e_2$	$b_{47} - e_3$
	e_5	$-b_{15} + e_4$	$-b_{25} + e_7$	$-b_{35} + e_6$	$-b_{45} - e_1$	$-a_5$	$b_{56} - e_3$	$b_{57} - e_2$
	e_6	$-b_{16} - e_7$	$-b_{26} + e_4$	$-b_{36} - e_5$	$-b_{46} - e_2$	$-b_{56} + e_3$	$-a_6$	$b_{67} + e_1$
	e_7	$-b_{17} + e_6$	$-b_{27} - e_5$	$-b_{37} - e_4$	$-b_{47} + e_3$	$-b_{57} + e_2$	$-b_{67} - e_1$	$-a_7$

Let L denote left translation by the vector $u = a + be_1 + ce_2 + de_3 + ee_4 + fe_5 + ge_6 + he_7$.

Then

$\det(L) = (a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2)^3 [a^2 + a_1b^2 + a_2c^2 + a_3d^2 + a_4e^2 + a_5f^2 + a_6g^2 + a_7h^2]$
independent of the b_{ij} 's. We have

Theorem 3. The modified octonion algebra in Table (2) is a division algebra if and only if $a_i > 0$ for $i = 1, \dots, 7$.

Theorem 4. The algebras represented by Table (2) are alternative if and only if all the a_i 's are 1 and all the b_{ij} 's are 0.

Proof. As before,

$$e_i(e_j e_j) = -a_j e_i \text{ and for some } k, (e_i e_j) e_j = (b_{ij} \pm e_k) e_j = b_{ij} \pm b_{ij} e_j - e_i.$$

Since the octonions are alternative, it follows that the algebras given by Table (2) are distinct from the octonions unless all the a_i 's are 1 and all the b_{ij} 's are 0.

I acknowledge L. D. Kugler's editorial assistance with this paper.

BIBLIOGRAPHY

- [1] A. CAYLEY, On the 8-square imaginaries, American Journal of Mathematics, 4, 293-296 (1881)
- [2] C. H. CURTIS, The four and eight square problem and division algebras, in Studies in Modern Algebra, Edited by A. A. Albert, Prentice Hall (1963).

SPECIAL RECTILINEAR CONGRUENCES

Gr. Tsagas, Ch. Christophoridou and As. Synefaki

School of Technology
Department of Mathematics and Physics
Aristotle University of Thessaloniki
Thessaloniki 540 06 - Greece

Abstract

The aim of the present paper is to determine surfaces in $\mathbf{R}^3$ whose two sheets of their evolute are degenerated into curves.

Keywords: surface, normal rectilinear, congruence, evolute, sheet, curve.

MSC (A.M.S.) 54A05

Reprinted from: Tsagas, Gr. T., Editor, “*New Frontiers in Algebras, Groups and Geometries*,” Hadronic Press, ISBN 1-57485-009-1, p. 317-326 (1996)

Copyright © 2026 by Hadronic Press Inc., Palm Harbor, FL 34682, U.S.A.

1. Introduction. Let S be a surface in Euclidean real manifold of three dimension $\mathbf{R}^3$.

We assume that S is covered only by a coordinate patch. All the normals to S form a fibre bundle over S , which is called normal bundle over S . It is known that there exist two surfaces S_1 and S_2 of $\mathbf{R}^3$ such that each normal to S is tangent to S_1 and S_2 . The surfaces S_1 and S_2 are called sheets of the evolute of S . The normal bundle of the surface S is a normal rectilinear congruence whose focal surfaces are the sheets of the evolute of S . The normals to S establish a one-to-one mapping between S_1 and S_2 under the condition that these are not degenerated.

The whole paper contains five paragraphs.

The introduction is contained in the first paragraph.

The second paragraph includes general theory of the sheets of the evolute of a given surface and the conditions under which the two sheets are degenerated into curves.

The study of surfaces of revolution, whose the sheets of their evolutes are degenerated into curves, is contained in the third paragraph.

The fourth paragraph has the determination of a surface, using the programme "MATHEMATICA" whose the sheets of its evolute are degenerated into curves.

The graphic representation of two special surfaces, each of them coming from two mentioned families of surfaces, is given at the last paragraph.

2. Let $\mathbf{R}^3$ be the real Euclidean manifold, which is referred to fixed rectangular coordinate system Oxyz. Therefore every surface S in $\mathbf{R}^3$, which is covered only by a coordinate patch, can be defined by the vector equation

$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k} \quad (2.1)$$

If $E, F, G; L, M, N$ are the fundamental magnitudes of the first and second order of S , whose parametric curves $u=\text{const.}$, $v=\text{const.}$ are the lines of curvature, then we have

$$F=M=0, \quad k_1=1/\rho_1=L/E, \quad k_2=1/\rho_2=N/G. \quad (2.2)$$

where k_1, k_2 are the principal curvatures of S . We assume that S is not a spherical surface and therefore $k_1, k_2 \neq 0$.

The Gauss equation and the Codazzi equations of the surface S take the form ([1], p.129)

$$LN = -\sqrt{EG} \left[\frac{\partial}{\partial u} \left(\frac{1}{\sqrt{E}} \frac{\partial \sqrt{G}}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{1}{\sqrt{G}} \frac{\partial \sqrt{E}}{\partial v} \right) \right] \quad (2.3)$$

$$\frac{\partial L}{\partial v} = \frac{1}{2} \left(\frac{L}{E} + \frac{N}{G} \right) \frac{\partial E}{\partial v}, \quad \frac{\partial N}{\partial u} = \frac{1}{2} \left(\frac{L}{E} + \frac{N}{G} \right) \frac{\partial G}{\partial u} \quad (2.4)$$

The normals to the surface S form a normal rectilinear congruence, whose focal surfaces are defined by the equations

$$S_1 \rightarrow \vec{R}_1 = \vec{r} + \rho_1 \vec{l}_0, \quad S_2 \rightarrow \vec{R}_2 = \vec{r} + \rho_2 \vec{l}_0 \quad (2.5)$$

where $\vec{l}_0$ is the unit normal vector field on S . This normal rectilinear congruence forms the normal bundle over S .

The fundamental magnitudes of the first order of the surfaces defined by equations (2.5) are given by ([18], pp. 151-152)

$$E_1 = \left(\frac{\partial \vec{R}_1}{\partial u} \right)^2 = \frac{\partial(\rho_1^2)}{\partial u}, \quad F_1 = \frac{\partial \vec{R}_1}{\partial u} \frac{\partial \vec{R}_1}{\partial v} = \frac{\partial \rho_1}{\partial u} \frac{\partial \rho_1}{\partial v}, \quad G_1 = \left(\frac{\partial \vec{R}_1}{\partial v} \right)^2 = \frac{\partial(\rho_1^2)}{\partial v} + G \left(1 - \frac{\rho_1}{\rho_2} \right)^2 \quad (2.6)$$

$$E_2 = \left(\frac{\partial \vec{R}_2}{\partial u} \right)^2 = \frac{\partial(\rho_2^2)}{\partial u} + E \left(1 - \frac{\rho_2}{\rho_1} \right)^2, \quad F_2 = \frac{\partial \vec{R}_2}{\partial u} \frac{\partial \vec{R}_2}{\partial v} = \frac{\partial \rho_2}{\partial u} \frac{\partial \rho_2}{\partial v}, \quad G_2 = \left(\frac{\partial \vec{R}_2}{\partial v} \right)^2 = \frac{\partial(\rho_2^2)}{\partial v} \quad (2.7)$$

In order the normal rectilinear congruence to have some additional properties, it is required the surface S to be special one. Below we study some special cases.

In order S_1 and S_2 are degenerated into curves we must have

$$\text{For } S_1: \frac{\partial \vec{R}_1}{\partial u} = 0 \Leftrightarrow \frac{\partial(p_1^2)}{\partial u} = 0 \Leftrightarrow p_1 = p_1(v) \quad (2.8)$$

$$\text{For } S_2: \frac{\partial \vec{R}_2}{\partial v} = 0 \Leftrightarrow \frac{\partial(p_2^2)}{\partial v} = 0 \Leftrightarrow p_2 = p_2(u) \quad (2.9)$$

From (2.8) we conclude $\vec{R}_1 = \vec{R}_1(v)$ and therefore S_1 is degenerated into a curve c_1 .

Similarly from (2.9) we have $\vec{R}_2 = \vec{R}_2(u)$ and hence S_2 is degenerated into a curve c_2 .

3. We assume that S is a surface of revolution. Then there exist a rectangular coordinate system $Oxyz$ and a special parametrisation on S such that it can be represented as follows

$$x = u \cos v, \quad y = u \sin v, \quad z = f(u) \quad (3.1)$$

The fundamental magnitudes of the first and second order of S are given by

$$E = 1 + f'^2, \quad F = 0, \quad G = u^2 \quad (3.2)$$

$$L = f''/(1 + f'^2)^{1/2}, \quad M = 0, \quad N = uf'/(1 + f'^2)^{1/2} \quad (3.3)$$

Therefore the radii of curvature, by means of (3.2) and (3.3), take the form

$$\rho_1 = E/L = (f'')^{-1}(1 + f'^2)^{3/2}, \quad \rho_2 = G/N = u(f')^{-1}(1 + f'^2)^{1/2} \quad (3.4)$$

From the second of (3.4) and the third of (2.7) we conclude that

$$\frac{\partial \vec{R}_2}{\partial v} = 0 \Rightarrow \vec{R}_2 = \vec{R}_2(u) \quad (3.5)$$

which means the focal surface S_2 of the evolute of S is degenerated into a curve.

If we have

$$\frac{\partial p_1^2}{\partial u} = \frac{\partial}{\partial u} \frac{(1 + f'^2)^3}{f''^2} = 0 \quad (3.6)$$

then we obtain the relation

$$(1+f'^2)^3 = cf''^2. \quad (3.7)$$

The (3.7) can be written

$$f''(1+f'^2)^{-3/2} = c_1$$

which by integration implies

$$\frac{f'^2}{1+f'^2} = (c_1 u + c_2)^2 \quad (3.9)$$

The equation (3.9) can be written

$$f' = \pm \frac{c_1 u + c_2}{\sqrt{1 - (c_1 u + c_2)^2}} \quad (3.10)$$

The equation (3.10) after some estimates implies

$$f = \pm \frac{1}{c_1} \sqrt{1 - (c_1 u + c_2)^2} + c_3. \quad (3.11)$$

From the above we obtain the following theorem.

Theorem 3.1 For every triple of constants (c_1, c_2, c_3) with $c_1 \neq 0$ and $1 - (c_1 u + c_2)^2 > 0$ corresponds a surface of revolution S represented by the equations (3.1), where the function f is given by (3.11).

4. Now, the following question can be put. Are there other surfaces whose the two sheets of their evolutes are degenerated into curves?

Using "MATHEMATICA" and checking eighty eight cases we have found one surface whose sheets of its evolute are degenerated into curves.

This surface is represented by parametric equations

$$x = \frac{u(q^2 + 2v^2)}{q^2 + u^2 + v^2}, \quad y = \frac{v(q^2 + 2u^2)}{q^2 + u^2 + v^2}, \quad z = \frac{q(u^2 - v^2)}{q^2 + u^2 + v^2} \quad (4.1)$$

where $q = \text{constant}$.

The fundamental magnitudes of the first and second order of this surface S are the following

$$E = \left(\frac{q^2 + 2v^2}{q^2 + u^2 + v^2} \right)^2, \quad F = 0, \quad G = \left(\frac{q^2 + 2u^2}{q^2 + u^2 + v^2} \right)^2 \quad (4.2)$$

$$L = \frac{2q(q^2 + 2v^2)}{(q^2 + u^2 + v^2)^2}, \quad M = 0, \quad N = \frac{-2q(q^2 + 2u^2)}{(q^2 + u^2 + v^2)^2} \quad (4.3)$$

The radii of the curvature of this surface take the form

$$\rho_1 = \frac{E}{L} = \frac{(q^2 + 2v^2)}{2q}, \quad \rho_2 = \frac{G}{N} = -\frac{(q^2 + 2u^2)}{2q} \quad (4.4)$$

From (3.15) we conclude that

$$\left(\frac{\partial \vec{R}_1}{\partial u} \right)^2 = E_1 = \frac{\partial}{\partial u} \left(\frac{(q^2 + 2v^2)^2}{4q^2} \right) = 0 \Rightarrow \vec{R}_1 = \vec{R}_1(v) \quad (4.5)$$

$$\left(\frac{\partial \vec{R}_2}{\partial v} \right)^2 = G_2 = \frac{\partial}{\partial v} \left(\frac{(q^2 + 2u^2)^2}{4q^2} \right) = 0 \Rightarrow \vec{R}_2 = \vec{R}_2(u) \quad (4.6)$$

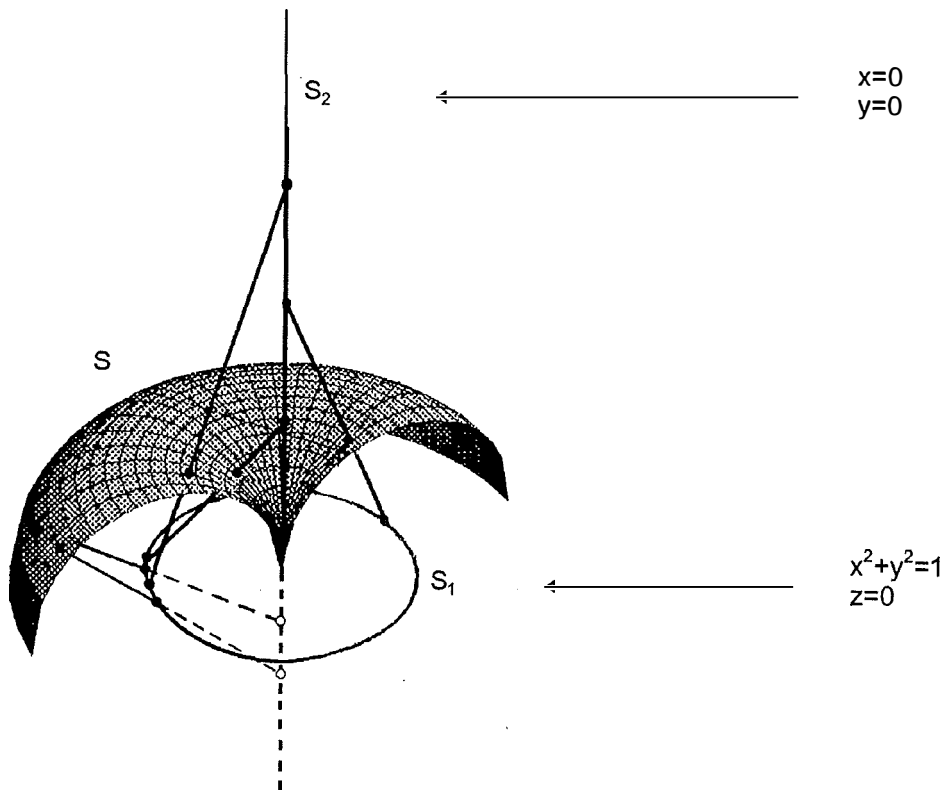
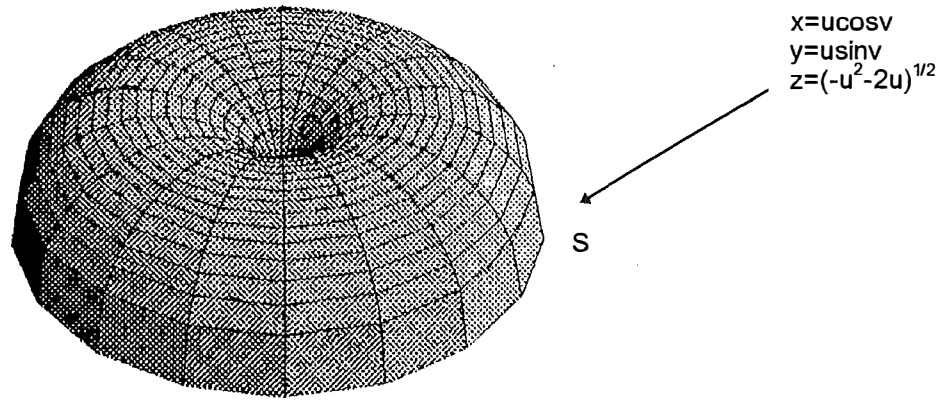
From the above we have obtained the following theorem.

Theorem 4.1 *The two sheets of the evolute of the surface S , which is represented by the equations (4.1) are degenerated into curves.*

5. The parametric equations of the surface of revolution (3.1), determined by the (3.11) and for the values of $c_1=c_2=1$ and $c_3=0$, take the form

$$\begin{aligned} x &= u \cos v \\ y &= u \sin v \\ z &= \pm(-u^2 - 2u)^{1/2}. \end{aligned}$$

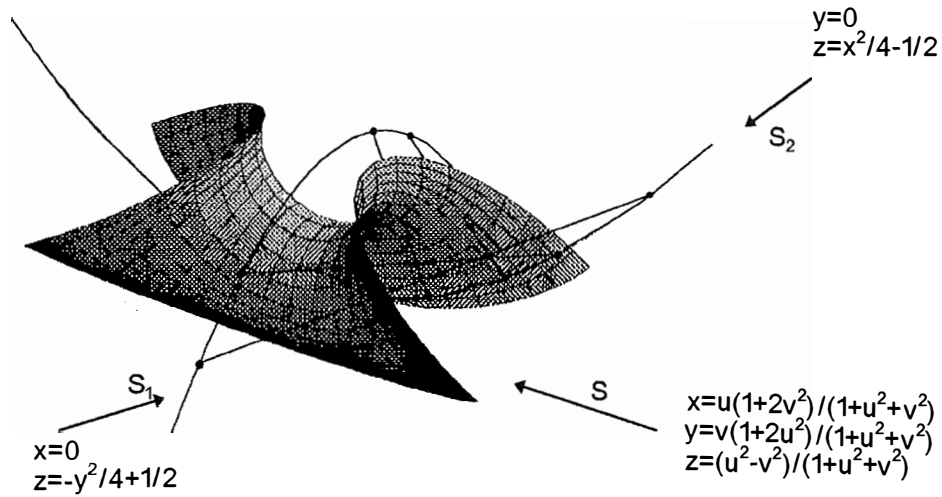
The graphic representation of the surface and for the sign + has the below shape. The curves S_1 and S_2 are the circle $z=0$, $x^2+y^2=1$ and the straight line $x=0$, $y=0$ respectively.



The parametric equations of the surface (4.1) and for the value $q=1$, take the form

$$\begin{aligned}x &= u(1+2v^2)/(1+u^2+v^2) \\ y &= v(1+2u^2)/(1+u^2+v^2) \\ z &= (u^2-v^2)/(1+u^2+v^2)\end{aligned}$$

The graphic representation of this surface has the below shape. The curves S_1 and S_2 are parabolas.



REFERENCES

- [1] Blaschke, W. : *Vorlesungen über Differentialgeometrie*, Band I, Berlin, Springer Verlag, 1924.
- [2] Caratheodory, C. : *Bemerkungen zu den Strahlenabbildungen der geometrischen Optik*, Math. Annalen, 144(1937).
- [3] Eingenharts, L.P. : *Introduction to differential geometry*, Princeton University Press, 1847.
- [4] Finicoff, S. : *Concruences avec les deux nappes de la surface focale applicables l' une sur l' autre par points correspondants*. Annali di Mat. 4 (1924), p. 175 à 184.
- [5] Granstein, W. : *Differential geometry*, New York. MacMillan, 1947.
- [6] Kommerell, V. and K. : *Theory der Raumkurven und Krummer Flächen*, I Band und II Band, W. de Gruyter, 1931.
- [7] Pylarinos, O. : *Sur certaine réseau de courbes tracées sur une surface*, Bul, Sc. Math. 2e series 84(1960), p.116 à 144.
- [8] Pylarinos, O. : *Sur les surfaces a courbure moyenne constante applicables sur des surfaces de révolution*, Annali di Mathematica pure ed applicata (IV), Vol. LIX(1962), pp. 319-350.
- [9] Schur, A. : *Über diejenigen Strahlensysteme, deren Brennflächen durch Systemstrahlen isometrisch aufeinander bezogen werden*, Math. Zeitschrift 19(1924), ss. 114-127.
- [10] Tsagas, Gr : *Two special categories of normal rectilinear congruences*, Bull. de la Soc. Math de Grece 7(1966), pp. 161-172.
- [11] Tsagas, Gr : *Sur deux classes particulieres de congruence de normales*, C.R, Acad. Sc. Paris, t. 263(1966), pp. 407-409.
- [12] Tsagas, Gr : *Classification of rectilinear congruences with special properties*. Tensor 26(1972), pp. 271-276.
- [13] Tsagas, Gr : *On the rectilinear congruences establishing an area preserving representation*. Tensor, N.S. Volume 29(1975), pp. 147-160.
- [14] Tsagas, Gr : *On the surfaces whose normal bundles have special properties*, Bull. Soc. Math. Grèce.Tome 15(1974), 59-67.

- [15] Tsagas, Gr: *On the rectilinear congruences whose straight lines are tangent to one parameter family of curves surface*. Tensor, N.S. Volume 29(1975), pp. 287-294.
- [16] Tsagas, Gr., Vougiouklis, Th. and Terzis, G. : *Some properties of the normal bundle of a special surface*. Mathematica Balkanica 7 (1977), 307-313.
- [17] Tsagas, Gr., Kessoglidis, M., Terzis, G. and Vougiouklis, Th. : *Special class of Raffy's surfaces. Proceedings of the summer school 1980 on applied Mathematics and Informatics for socio-economic development*. Thessaloniki 1980, 315-320.
- [18] Vessiot, L. : *Leçon de géométrie supérieure*, Paris, Librairie Scientifique, J.Herman, 1972.
- [19] Heatherburn, G. : *Differential Geometry*, Vol (I) and (II) Cambridge University press, (1931).

WALL MONOIDS I

I. G. Rosenberg

Mathématiques et Statistique
Université de Montréal, C. P. 6128
Succ. Centre-ville Montréal, Qué.
Canada H3C 3J7
rosenb@ere.umontreal.ca

Abstract

In 1937 Wall generalized the notion of a (Marty) hypergroup on H by stipulating that its values are submultisets of H rather than mere subsets of H . In 1991 Corsini and Mittas further extended this by allowing infinite frequencies and by removing some of the restrictions. We study this in more depth; in particular the associativity provided each multiset has exactly one nonzero frequency. Given a monoid the determination of these frequencies such that the resulting Wall monoid is associative seems to be a genuine problem. We do this for the additive monoid of nonnegative integers and for the cyclic groups $\mathbb{Z}_k$, $k > 2$.

Acknowledgments: The author gratefully acknowledges support from the Università degli Studi di Udine, the CNR, Italy, and the NSERC, Canada.

Reprinted from: Thomas Vougiouklis., Editor, “*New Frontiers in Hyperstructures*,”
Hadronic Press, ISBN 1-57485-008-3, p. 159-166 (1996)

Copyright © 2026 by Hadronic Press Inc., Palm Harbor, FL 34682, U.S.A.

I. G. ROSENBERG

0. INTRODUCTION

A (Marty) hypergroupoid on a set H is given by a multiplication or Cayley table whose entries are nonvoid subsets of H (not necessarily singletons). The most studied among them are semihypergroups which are hypergroupoids satisfying a natural associative law. Hypergroups are semihypergroups in which H is the union of the entries in every row as well as in each column. Hypergroups are more much general structures than groups; e.g., even for H of a small prime cardinality p there is a very large number of nonisomorphic hypergroups while up to isomorphism but one group $\mathbb{Z}_p$. Very early H. S. Wall introduced structures in which the values are no longer subsets of H but multisubsets of H . Here a multisubset of H is an assignment of nonnegative integer frequencies to the elements of H . It seems that for 55 years Wall's idea was not followed. Only in 1991 Corsini and Mittas generalized his definition by allowing infinite-cardinal frequencies and by removing the constant sum of frequencies stipulation. To show the range of problems involved they considered the very special case of the groups $\mathbb{Z}_2, \mathbb{Z}_3$ and $\mathbb{Z}_4$ in which for every pair (x, y) the sum $x \oplus y$ has a positive frequency a_{xy} while all the other elements of $\mathbb{Z}_i$ have frequency 0.

In this paper we carry the research a little bit further. As it usual in algebra, we are mostly interested in multisets with finite frequencies. In this context, the associative law for singletons in general does not imply the associativity for arbitrary multisets. We show that the general associative law is guaranteed in special semihypergroups which we call profinite.

§1 consists of definitions and results along these lines. In §2 we give some simple results for W -monoids which are defined in the same way as the above Corsini–Mittas examples as monoids with frequencies. In §3 we characterize all commutative W -monoids over the additive monoid of nonnegative integers. They are singled out by rather delicate divisibility conditions on the frequencies assigned to the pairs $(1, i)$. Finally, in §4 we derive from this result a characterization of the W -monoids over the cyclic group $\mathbb{Z}_k (k = 2, 3, \dots)$. 2.1

§1 W -HYPERGROUPOIDS AND W -SEMIHYPERGROUPS

Recall the definition of (Marty) hypergroupoid (see [Co91]). A *hypergroupoid* is a pair $\mathcal{H} = (H; \circ)$ where H is a set and $(x, y) \mapsto x \circ y$ is a map from H^2 into the set $\mathcal{P}^*(H)$ (of all nonvoid subsets of H). The hyperoperation $\circ$ is extended to $\mathcal{P}^*(H)$ by setting

$$\xi \circ \eta := \bigcup_{a \in \xi} \bigcup_{b \in \eta} a \circ b \quad (1)$$

for all $\emptyset \neq \xi, \eta \subseteq H$. (For example, the left side $x \circ (y \circ z)$ of the associative law is (1) for $\xi := \{x\}$ and $\eta := y \circ z$).

In the definition of $\mathcal{H}$ we can replace the set $x \circ y$ by its characteristic function $\chi_{x \circ y}$ (i.e., $\chi_{x \circ y}(a) = 1$ if $a \in x \circ y$ and $\chi_{x \circ y}(a) = 0$ otherwise). Now “ $\circ$ ” can be perceived

WALL MONOIDS I

as a map from H^2 into the set X_2 of all maps $H \longrightarrow \{0, 1\}$ distinct from the constant 0-valued map. Then (1) becomes

$$\chi_{\xi \circ \eta}(h) = \max_{x, y \in H} \{\chi_{\xi}(x) \chi_{\eta}(y) \chi_{x \circ y}(h)\} \quad (2)$$

for all $h \in H$. In 1937 H. S. Wall [Wa37] proposed to replace X_2 by the set of all maps $\chi : H^2 \longrightarrow \{0, 1, \dots, n\}$ such that $\sum_{h \in H} \chi(h) = n$ (where n is a fixed positive integer); clearly each $x \circ y$ can be viewed as a multiset on H (in which each $h \in H$ appears with the frequency $\chi(h)$). This was further extended by Corsini and Mittas in 1991 [CoMi91]. Their structure can be defined as follows. Let α be an infinite regular cardinal greater than the cardinality of H and let C_{α} denote the set of all cardinals smaller than α . Let Y_{α} denote the set of all maps $H \longrightarrow C_{\alpha}$ distinct from the constant 0-map. A **W-hypergroupoid of type α** is $\mathcal{H} = \langle H; \circ \rangle$ where $(x, y) \mapsto x \circ y$ is a map from H^2 into Y_{α} . The hyperoperation $\circ$ is extended to Y_{α} as follows: For all $\xi, \eta \in Y_{\alpha}$ the map $\xi \circ \eta$ from H into Y_{α} is defined by setting for all $h \in H$

$$(\xi \circ \eta)(h) := \sum_{i, j \in H} \xi(i) \cdot \eta(j) \cdot (i \circ j)(h) \quad (3)$$

(here the sum and the products are in the cardinal arithmetic; e.g., $a \cdot b$ is the usual product if a and b are finite and $a \cdot b = \max(a, b)$ otherwise).

We are interested in a finitary version of the definition. Set $F := Y_{\aleph_0}$ (i.e. F is the set of all maps from H to the $\mathbb{N}$ (= the set $\{0, 1, \dots\}$ of all nonnegative integers) with nonvoid support. A **W-hypergroupoid** is $\mathcal{H} = \langle H; \circ \rangle$ where $(x, y) \mapsto x \circ y$ is a map from H^2 into F . A **W-hypergroupoid $\mathcal{H}$ is a W-semihypergroup** if for all $a, b, c \in H$

$$(\chi_a \circ \chi_b) \circ \chi_c = \chi_a \circ (\chi_b \circ \chi_c) \quad (4)$$

where χ_a stands for the above characteristic function $\chi_{\{a\}} : H \longrightarrow \{0, 1\}$ and the products are calculated according to (3)). For notational simplicity we write (4) as $(a \circ b) \circ c = a \circ (b \circ c)$. In a semihypergroup or even semigroup the product of k elements can be written without parentheses as $a_0 a_1 \dots a_k$. We would like have this facility in W-semihypergroups as well but, in general, there is a finiteness problem since there is no guarantee that $(a \circ b) \circ c$ belongs to F (i.e. is a map from H to $\mathbb{N}$) and the same holds for more complex products. This leads to the following restriction. Set $H^{(1)} := \{\chi_h : h \in H\}$ and for $n \in \mathbb{N}, n > 1$ define $H^{(n)}$ recursively by

$$H^{(n+1)} := \{\xi \circ \eta : \xi \in H^{(k)}, \eta \in H^{(l)} \text{ for some } k, l \in \mathbb{N}^*, k + l = n + 1\}.$$

Set $H^{(\circ)} := \bigcup_{n \in \mathbb{N}^*} H^{(n)}$. A **W-hypergroupoid is profinite** if $H^{(\circ)} \subseteq F$. From (3) we obtain the following two sufficient profiniteness conditions :

(i) **If the relations**

$$\rho_h := \{(x, y) \in H^2 : (x \circ y)(h) > 0\} \quad (5)$$

$(h \in H)$ are finite (i.e. $|\rho_h| < \aleph_0$ for all $h \in H$) or (ii) if $\text{supp}(i \circ j) := \{h \in H : (i \circ j)(h) > 0\}$ is finite for all $i, j \in H$, then $\langle H; \circ \rangle$ is profinite.

Notice that for a profinite W -hypergroupoid $\langle H; \circ \rangle$ and $\xi, \eta \in H^{(\circ)}$ the sum and the product in (3) are just the arithmetical ones. We show that in a profinite W -semihypergroup the associative law $(\xi \circ \eta) \circ \zeta = \xi \circ (\eta \circ \zeta)$ holds for all $\xi, \eta, \zeta \in H^{(\circ)}$.

Proposition 1.1. The following are equivalent for a profinite W -hypergroupoid $\mathcal{H} = \langle H; \circ \rangle$:

- (i) $\mathcal{H}$ is a W -semihypergroup,
- (ii)

$$\sum_{i \in H} (a \circ b)(i)(i \circ c)(h) = \sum_{j \in H} (b \circ c)(j)(a \circ j)(h) \quad (6)$$

holds for all $a, b, c, h \in H$, and

$$(iii) \quad (\xi \circ \eta) \circ \zeta = \xi \circ (\eta \circ \zeta) \quad (7)$$

holds for all $\xi, \eta, \zeta \in H^{(\circ)}$.

Proof. (i) $\implies$ (ii). Let (4) hold and let $a, b, c, h \in H$. According to (3) and the fact that $\chi_x(y) = 0$ for all $y \neq x$ and $\chi(x) = 1$

$$\begin{aligned} (\chi_a \circ \chi_b) \circ \chi_c(h) &= \sum_{i,j} \left(\sum_{k,l} \chi_a(k) \chi_b(l) (k \circ l) \right) \chi_c(j) (i \circ j)(h) \\ &= \sum_i (a \circ b)(i)(i \circ c)(h) \end{aligned}$$

(all sums are over H and are finite because H is profinite). Similarly, $\chi_a \circ (\chi_b \circ \chi_c)(h)$ is the right hand side of (6).

(ii) $\implies$ (iii). Let (6) hold for all $a, b, c, h \in H$ and let $\xi, \eta, \zeta \in H^{(\circ)}, h \in H$. By (3) and (6)

$$\begin{aligned} ((\xi \circ \eta) \circ \zeta)(h) &= \sum_{j,k,l} \xi(k) \eta(l) \zeta(j) \sum_i (k \circ l)(i)(i \circ j)(h) \\ &= \sum_{j,k,l} \xi(k) \eta(l) \zeta(j) \sum_i (l \circ j)(i)(k \circ i)(h) \\ &= \sum_{k,i} \xi(k) \left(\sum_{l,j} \eta(l) \zeta(j) (l \circ j)(i) \right) (k \circ i)(h) \\ &= \sum_{k,i} \xi(k) (\eta \circ \zeta)(i) (k \circ i)(h) = \xi \circ (\eta \circ \zeta)(h). \end{aligned}$$

(iii) $\implies$ (i). Let $a, b, c, h \in H$. Setting $\xi := \chi_a, \eta := \chi_b$ and $\zeta := \chi_c$ in (7) we obtain (4).

□

§2. W-SEMIGROUPS

The remainder of this paper is devoted to profinite W-semigroups arising from semigroups in a very simple way. A W-hypergroupoid $\mathcal{H} = \langle H; \circ \rangle$ is a **W-groupoid** if $\text{supp } x \circ y$ is a singleton for all $x, y \in H$. Clearly $\mathcal{H}$ is a W-groupoid if and only if there exist a groupoid $\langle H; \cdot \rangle$ and positive integers a_{xy} ($x, y \in H$) such that $x \circ y = a_{xy} \chi_{x \cdot y}$ holds for all $x, y \in H$; in other words, a W-groupoid is a groupoid in which each product $x \cdot y$ has a frequency a_{xy} . We shall denote it by $\langle H; \cdot; A \rangle$ where $A = [a_{xy}]_{x, y \in H}$ is a (possibly infinite) positive integer matrix. A W-groupoid $\mathcal{H}$ is a **W-semigroup** if $\mathcal{H}$ is a semihypergroup.

Lemma 2.1. A W-groupoid $\mathcal{H} = \langle H; \cdot; A \rangle$ is a W-semigroup if and only if

- (i) $\langle H; \cdot \rangle$ is a semigroup, and
- (ii) for all $x, y, z \in H$

$$a_{xy} a_{x \cdot y, z} = a_{yz} a_{x, y \cdot z}. \quad (8)$$

Proof. Let $x, y, z \in H$. By (4)

$$(\chi_x \circ \chi_y) \circ \chi_z = a_{xy} a_{x \cdot y, z} \chi_{(x \cdot y) \cdot z}; \quad \chi_x \circ (\chi_y \circ \chi_z) = a_{x, y \cdot z} a_{yz} \chi_{x \cdot (y \cdot z)}.$$

□

By Lemma 2.1 every W-semigroup arises from a semigroup but for a given semigroup $\langle H; \cdot \rangle$ the characterization of all A satisfying (8) is not transparent. Of course, if all entries of A are equal, then (8) holds but already for the semigroups $\langle \mathbb{N}; + \rangle$ and $\langle \mathbb{Z}_k; \oplus \rangle$ ($k > 1$) we obtain somewhat weird divisibility conditions. We start with some easy general conditions for monoids and abelian semigroups.

Lemma 2.2. Let $\langle G; \cdot; A \rangle$ be a W-semigroup. If $e \in G$ is a left (right) neutral element of $\langle G; \cdot \rangle$ then $a_{ex} = a_{ee}(a_{xe} = a_{ee})$ for all $x \in G$. If $\langle G; \cdot; e \rangle$ is a monoid then $a_{ex} = a_{ee} = a_{xe}$ holds for all $x \in G$.

Proof. Let e be left neutral element of $\langle G; \cdot \rangle$. Let $x \in G$. Applying (8) to the triple (e, e, x) we obtain $a_{ee} a_{ex} = a_{ex} a_{ee}$. In view of $a_{ex} > 0$ this yields $a_{ex} = a_{ee}$. The proof for the right neutral element is similar and the last statement is just the combination of the preceding ones. □

The next lemma provides a pullback from a W-semigroup under a semigroup homomorphism. For $f : G \rightarrow H$ and $A = [a_{xy}]_{x, y \in H}$ set $A^{(f)} := [a_{f(x)f(y)}]_{x, y \in G}$.

Lemma 2.3. Let $\mathcal{G} = \langle G; \cdot \rangle$ and $\mathcal{H} = \langle H; \circ \rangle$ be semigroups, let $f : x \mapsto x'$ be a homomorphism from $\mathcal{G}$ into $\mathcal{H}$ and let $A = [a_{xy}]_{x, y \in H}$ be a matrix over $\mathbb{N}^*$. Then

- (i) If $\langle H; \cdot; A \rangle$ is a W-semigroup then $\langle G; \cdot; A^{(f)} \rangle$ is a W-semigroup, and
- (ii) Let f be a surjection. Then $\langle H; \circ; A \rangle$ is a W-semigroup if and only if $\langle G; \cdot; A^{(f)} \rangle$ is a W-semigroup.

Proof. Set $A' := [a'_{xy}]_{x,y \in G} = A^{(f)}$. (i) To verify (8) for A' let $x, y, z \in G$. Then

$$a'_{xy} a'_{x,y,z} = a_{x'y'} a_{(xy)',z'} = a_{x'y'} a_{x'o'y',z'} = a_{y'z'} a_{x',y'o'z'} = a_{y'z'} a_{x',(y.z)'} = a'_{yz} a'_{x,y,z}$$

and so $\langle G; \cdot; A' \rangle$ is a W-semigroup.

(ii) Let f be surjective. ($\implies$) The statement (i). ($\impliedby$) Let $\langle G; \cdot; A' \rangle$ be a W-semigroup. To prove (8) for A let $x, y, z \in H$. As f is surjective, $x = u', y = v'$ and $z = w'$ for some $u, v, w \in G$. Now

$$\begin{aligned} a_{xy} a_{xoy,z} &= a_{u'v'} a_{u'ov',w'} = a_{u'v'} a_{(u.v)',w'} = \\ a'_{uv} a'_{u.v,w} &= a'_{uw} a'_{u,v.w} = a_{v'w'} a_{u',(v.w)'} = \\ a_{v'w'} a_{u',v'ow'} &= a_{yz} a_{x,y,z} \end{aligned}$$

and so $\langle H; \circ; A \rangle$ is a W-semigroup. $\square$

The following lemma is evident.

Lemma 2.4. Let $\langle G; \cdot; [a_{xy}]_{x,y \in G} \rangle$ be a W-semigroup. If $\langle H; \cdot \rangle$ is a subsemigroup of $\langle G; \cdot \rangle$ then $\langle H; \cdot; [a_{xy}]_{x,y \in H} \rangle$ is a W-semigroup. $\square$

§3. THE W-MONOIDS OVER $\langle \mathbb{N}; + \rangle$

In this section we characterize W-monoids over the additive monoid of nonnegative integers. Such a monoid is of the form $\mathcal{N} = \langle \mathbb{N}; +; A \rangle$ where $A = [a_{xy}]_{x,y \in \mathbb{N}}$ is an infinite symmetric matrix over $\mathbb{N}^*$. For every $h \in \mathbb{N}$ the relation ρ_h defined in (5) is $\rho_h = \{(x, y) \in \mathbb{N}^2 : x + y = h\}$; whence $|\rho_h| = h + 1 < \aleph_0$ and so $\mathcal{N}$ is profinite. We characterize W-monoids $\mathcal{N}$ in terms of $\alpha_i := a_{1i}$ ($i \in \mathbb{N}$).

Proposition 3.1. $\mathcal{N} = \langle \mathbb{N}; +; A \rangle$ is a W-monoid if and only if

- (i) $a_{0i} = \alpha_0$ for all $i \in \mathbb{N}$ and
- (ii) for all $m \geq n > 1$

$$a_{mn} = \alpha_m \dots \alpha_{m+n-1} / (\alpha_1 \dots \alpha_{n-1}). \quad (9)$$

Proof. ($\implies$) Let $\mathcal{N}$ be W-monoid. The condition (i) follows from Lemma 2.1. To prove (ii) let (P_n) denote the statement “(ii) holds for all $m \geq n$ ”. We prove (P_n) by induction on $n = 2, 3, \dots, 1$. To prove (P_2) consider the triple $(1, 1, m)$. From (8) we obtain $a_{11} a_{2m} = a_{1m} a_{1,m+1}$; i.e., $\alpha_1 a_{2m} = \alpha_m \alpha_{m+1}$ and (P_2) holds. 2) Suppose that $n \geq 2$ and (P_n) holds. Apply (8) to the triple $(1, n, m)$ to obtain $a_{1n} a_{n+1,m} = a_{nm} a_{1,m+n}$; i.e., $\alpha_n a_{n+1,m} = \alpha_{m+n} a_{nm}$. Substitute the right-hand of (9) for a_{nm} :

$$\alpha_n a_{n+1,m} = \alpha_{n+m} \alpha_m \dots \alpha_{m+n-1} / (\alpha_1 \dots \alpha_{n-1}).$$

Thus (P_{n+1}) holds. This concludes the inductive step and the proof of necessity.

($\impliedby$) Let (i) and (ii) hold. Let $x, y, z \in \mathbb{N}$. If $x = 0$ then from (i) we obtain

WALL MONOIDS I

$a_{0y} \cdot a_{yz} = a_{yz} \cdot a_{y+z,0}$ and so (8) holds. A similar argument can be used if $y = 0$ or $z = 0$. Thus let $x, y, z > 0$. Notice that (8) is trivially valid if $m = 1$ or $n = 1$ (whereby, as usual, for $n = 1$ the denominator in (9) is understood to be 1). Applying (9) several times we obtain

$$\begin{aligned} a_{xy} a_{z, x+y} &= \frac{\alpha_y \dots \alpha_{x+y-1}}{\alpha_1 \dots \alpha_{x-1}} \cdot \frac{\alpha_{x+y} \dots \alpha_{x+y+z-1}}{\alpha_1 \dots \alpha_{z-1}} = \frac{\alpha_y \dots \alpha_{y+z-1}}{\alpha_1 \dots \alpha_{z-1}} \cdot \frac{\alpha_{y+z} \dots \alpha_{x+y+z-1}}{\alpha_1 \dots \alpha_{x-1}} \\ &= a_{yz} a_{x, y+z} \end{aligned}$$

Thus (8) holds and so $\mathcal{N}$ is a W -monoid. $\square$

Example 3.2. Let $c, k, l \in \mathbb{N}^*$. Set $a_{0i} := c$ for all $i \in \mathbb{N}$ and for all $n \in \mathbb{N}^*$ put $\alpha_{2n-1} := k$ and $\alpha_{2n} := l$. Then for all $m, n \in \mathbb{N}^*$

$$a_{mn} = \begin{cases} k & \text{if } m+n \text{ is even,} \\ l & \text{otherwise.} \end{cases}$$

The (infinite) matrix A is

$$\begin{bmatrix} c & c & c & c & c & \dots \\ c & k & l & k & l & \dots \\ c & l & k & l & k & \dots \\ c & k & l & k & l & \dots \\ c & l & k & l & k & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

We reformulate Proposition 3.1.

Corollary 3.3. Let $\langle \alpha_i : i \in \mathbb{N} \rangle$ be a sequence over $\mathbb{N}$. There exists a W -monoid $\langle \mathbb{N}; +; A \rangle$ with $a_{1i} = \alpha_i$ for all $i \in \mathbb{N}$ if and only if for all $m \geq n > 1$ the number $\alpha_1 \dots \alpha_{n-1}$ divides $\alpha_m \dots \alpha_{m+n-1}$. If this condition is satisfied then the unique symmetric matrix A is determined by (9). $\square$

§4. THE W -MONOIDS OVER $\langle k; \oplus \rangle$

For $k > 1$ denote by $\oplus$ the sum mod k over $k := \{0, 1, \dots, k-1\}$ (i.e., $x \oplus y = x + y$ if $x + y < k$ and $x \oplus y = x + y - k$ otherwise) and set $\mathbb{Z}_k := \langle k; \oplus \rangle$. As it is well known, $\mathbb{Z}_k$ is the cyclic k -element group. The following characterization of the commutative W -monoids over $\mathbb{Z}_k$ is derived from Proposition 3.1.

For every $m \in \mathbb{N}$ set $\alpha_m := a_{1n}$ where $n \in k$ satisfies $n \equiv m \pmod{k}$.

Proposition 4.1. Let $k \geq 2$. Then $\langle k; \oplus; A \rangle$ is a commutative W -monoid if and only if

- (i) $a_{00} = \dots = a_{0, k-1} = \alpha_0$, and
- (ii) for all $k > m \geq n > 1$

$$a_{mn} = \alpha_m \dots \alpha_{m+n-1} / (\alpha_1 \dots \alpha_{n-1}).$$

Proof. Proposition 3.1 and Lemma 2.2(ii) (where x' is the element of k congruent x mod k). $\square$

Remark. For $k = 2, 3, 4$ the statement of Proposition 4.1 is essentially in [CoMi 92] (examples following Thm 3).

Example. $\langle 5; \oplus; A \rangle$ is a commutative W-monoid if and only if A is a symmetric 5×5 matrix over $\mathbb{N}^*$ of the form

$$\begin{bmatrix} \alpha_0 & \alpha_0 & \alpha_0 & \alpha_0 & \alpha_0 \\ - & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ - & - & \alpha_2\alpha_3/\alpha_1 & \alpha_3\alpha_4/\alpha_1 & \alpha_4\alpha_0/\alpha_1 \\ - & - & - & \alpha_3\alpha_4\alpha_0/(\alpha_1\alpha_2) & \alpha_4\alpha_0/\alpha_2 \\ - & - & - & - & \alpha_4\alpha_0/\alpha_3 \end{bmatrix}$$

In other words, A determines a commutative W-monoid if and only if $\alpha_0, \dots, \alpha_4 \in \mathbb{N}^*$ satisfy

- a) α_1 divides the $\gcd(\alpha_2\alpha_3, \alpha_3\alpha_4, \alpha_4\alpha_0)$,
- b) $\text{lcm}(\alpha_2, \alpha_3)$ divides $\alpha_0\alpha_4$, and
- c) $\alpha_1\alpha_2$ divides $\alpha_0\alpha_3\alpha_4$.

For example, let $\alpha_0, \dots, \alpha_4$ be a given by the following table:

Then α_2 divides $12 = \gcd(12, 24, 24) = \gcd(\alpha_2\alpha_3, \alpha_3\alpha_4, \alpha_4\alpha_0)$, $\text{lcm}(\alpha_2, \alpha_3) = 12$ divides $24 = \alpha_4\alpha_0$ and $\alpha_1\alpha_2 = 8$ divides $3 \cdot 3 \cdot 4 = 36 = \alpha_0\alpha_3\alpha_4$. The matrix A filled in according to (i) and (ii) is

$$\begin{bmatrix} 3 & 3 & 3 & 3 & 3 \\ 3 & 2 & 4 & 3 & 8 \\ 3 & 4 & 6 & 12 & 12 \\ 3 & 3 & 12 & 9 & 6 \\ 3 & 8 & 12 & 6 & 8 \end{bmatrix}.$$

REFERENCES

- [Co 91] P. Corsini, *Prolegomena of hypergroup theory*, Suppl. **Revista di mat. pura et appl.**, Aviani Editore, Tricesimo, Italy, 1991, 214 pp..
- [CoMi 92] ——— and J. Mittas, *New topics of hypergroup theory*, Preprint Udine 1991, 23pp., To appear in **Multiple Valued Logic Int. J.**.
- [Wa 37] H. S. Wall, *Hypergroups*, **Amer. J. Math.** (1937), 77–98.

**INFINITE-DIMENSIONAL HOLOMORPHY WITHOUT
CONVEXITY CONDITION. I: THE LEVI PROBLEM
IN NON LOCALLY CONVEX SPACES**

Aboubakr Bayoumi
King Saud University
College of Science
Department of Mathematics
P. O. Box 2355
Riyadh 11451, Saudi Arabia

Abstract

In this survey article we study the solution of the Levi problem in different classes of non locally convex spaces and discuss the different approaches used to solve it. This means that we have shown the richness of the class of holomorphic functions in many non locally convex spaces.

Reprinted from: Tsagas, Gr. T., Editor, “*New Frontiers in Algebras, Groups and Geometries*,” Hadronic Press, ISBN 1-57485-009-1, p. 287-306 (1996)

Copyright © 2026 by Hadronic Press Inc., Palm Harbor, FL 34682, U.S.A.

Introduction

The linear and topological properties of several classes of non-locally convex spaces have received much attention in recent years. However the holomorphic properties of such spaces have been neglected.

The present survey is an attempt to introduce some of the fundamental holomorphic properties that have been obtained during the last decade.

We shall try in this article to explain and discuss the fact that, there are many holomorphic functions on different classes of non-locally convex spaces (e.g. F -spaces and locally pseudocovex spaces with bases) which are rich enough to allow a theory of holomorphic functions. In fact, this area which we may call **"infinite-dimensional Holomorphy without convexity condition"** has developed in the last few years [cf. 2,3,4,5,6,7].

However, our aim here is to focus attention on one of the fundamental and important problem in this field, namely the Levi problem. That is to prove that a pseudoconvex domain is a domain of holomorphy.

We shall discuss the different approaches we have used to solve the Levi problem in complete metric vector spaces (F-spaces) and in locally pseudocovex spaces (Lps) wherever they have finite dimensional Schauder decomposition (f.d. decomposition) or the bounded approximation property (b.a.p). see [2,3,4]. Thus essentially there is no new results in this survey.

It is natural to get, as a consequence of having a solution to the Levi problem for these classes of spaces, several other results in Functional and Complex Analysis. In fact this work has been inspired by several earlier results: [cf. 2,3,4,5,6,7,8,9,10,17,18,19,26,27].

For the sake of clarity we will consider only general topological vector spaces (tvs) E which are not assumed to be locally convex. For an open subset U of E , a function $f: U \rightarrow \mathbb{C}$ is said to be **holomorphic** if it is continuous and Gâteaux analytic. The open subset U is called **pseudoconvex** if $-\log S_U(x,a)$ is plurisubharmonic in $U \times E$, where $S_U(x,a) = \inf(\lambda \geq 0; x + \lambda a \in U)$, $x \in U$, $a \in E$, $t \in \mathbb{C}$, that is, for every finite dimensional subspace F of E , $(U \cap F)$ is pseudocovex in F . By a **plurisubharmonic** function we mean a function which is upper semi-continuous and subharmonic on every complex line. We call the open subset U as a domain of existence if there exists an $f \in H(U)$ which cannot be continued beyond the boundary ∂U . If the Levi problem has a solution for E , then the space E is called a **levi space**.

1- The Levi problem in PB-spaces and radius of convergence approach

In this section, we discuss the problem of constructing a holomorphic function with a given radius of convergence and its relation to the Levi problem for certain metric vector spaces containing locally bounded F-spaces.

Let U be an open set in a complex metric vector space E with a translation invariant metric d . It has been noticed that $\log d_U(x)$ is not necessarily plurisubharmonic whenever U is pseudocovex, where $d_U(x) = \inf_{y \in \partial U} d(x,y)$ is the distance to the boundary ∂U of U . This was one of the reasons to classify the metric vector spaces as follows: E is said to have the **pseudoconvex-boundary-distance property** (in short: the **PB-property**) if for every pseudoconvex domain U in E , the function $x \mapsto -\log d_U(x)$ is plurisubharmonic in U . Such a space is also called a **PB-space**. That a p -normed space E is a PB-space is easily seen from the definition of pseudoconvexity using the direction boundary distance $d(x,y) = \|x-y\|_p$ where $\|tx\| = |t|^p \|x\|$.

The following example shows that a PB-metric could be a non-homogeneous one.

Example 1. The space $\ell_p^- = \bigcup_{q \leq p} \ell_q = \{x \in \ell_p; x \in \ell_q \text{ for some } q \leq p\}$ with the metric $d(x,0) = \inf_q d_q(x,0) = \inf_q \sum_{j=1}^{\infty} |x_j|^q$ ($q \leq 1$) is a PB-space. Note that if U is pseudoconvex in ℓ_p^- , then $-\log d_U(x) = \sup_q -\log d_{qU}(x) = \sup_q \sup_y (-\log \sum_{j=1}^{\infty} |x_j - y_j|^q)$ is plurisubharmonic as supremum of plurisubharmonic functions, of course d is not p -homogeneous for any p , ($0 < p \leq 1$).

Further, the following proposition gives a sufficient condition on E to be a PB-space; see the author [2, proposition 2.4].

Theorem 1.1 Let d be a translation invariant metric on a complex metric space E such that the function

$$t \mapsto -\log d(e^t x, 0), \quad t \in \mathbb{C}$$

is convex in $\mathbb{C}$ for each fixed $x \in E$. Then (E, d) is a PB-space. In particular, this is the case if d is given by a p -homogeneous norm.

As an application of the above theorem we get

Example 2: A complex vector space E with the metric given by $d(x, y) = \frac{\|x-y\|}{1+\|x-y\|}$, where $\|\cdot\|$ is p -homogeneous norm, ($0 < p \leq 1$) is a PB-space. We note that the function

$$-\log d(e^t x, 0) = \log(1+\|e^t x\|) - \log\|e^t x\| = \log(1+\alpha e^{-tp}), \quad \alpha = \frac{1}{\|x\|}$$

is convex in $t \in \mathbb{C}$ for every fixed $x \in E$ with the standing of the fact that d is not p -homogeneous.

The radius of convergence $R_f(x)$ of a function $f \in H(U)$, $U \subseteq E$ is open in E , at a point $x \in U$ is defined as follows:

$R_f(x) = \sup\{r > 0; \text{ the Taylor series of } f \text{ at } x \text{ converges uniformly in } B(x, r)\}$. If $R_f^b(x)$ is the radius of boundedness of $f \in H(U)$ at x , i.e. $R_f^b(x) = \sup\{r > 0; \|f\|_{B(x, r)} \text{ is finite, } B(x, r) \subseteq U\}$, $x \in U$. Then it is easy to prove from the definition that, $R_f^b = \inf(R_f, d_U)$ see the author [2]. This extends Nachbin's result on normed spaces.

We have proved in [2; proposition 2.3, lemma 2.2] that the radius of convergence $R_f(x)$, $f \in H(U)$, has the following properties.

- (1) $-\log R_f$ is plurisubharmonic (if d is a PB-metric) and $R_f \leq d_U$.
 (2) $|R_f(x) - R_f(y)| \leq d(x, y)$ if $x, y \in U$ and $d(x, y) \leq \sup\{d_U(x), d_U(y)\}$
 (for normed spaces we refer to Lelong [21], Kiselman [17] and Schottenloher [27]).

Kiselman [17] has asked whether the radius of convergence R_f of a holomorphic function f on a normed space can be prescribed arbitrarily. It seems that his question has been inspired by the existence of an entire function f on a Banach space E with a very small radius of convergence, which is studied by Aron [1]. The following example is given by Kiselman [16]. It shows that there is no exact solution to this problem for the most sequence normed spaces.

Example 3. The function $R(x) = (1 + \|x\|)^{-1}$ on ℓ_p ($\infty > p \geq 1$) satisfies the above conditions 1), 2) but there is no $f \in H(\ell_p)$ with $R_f = R$, see Kiselman [16] for the approximate solution in normed spaces.

One of the interesting results in infinite-dimensional holomorphy in normed spaces E is given by Schottenloher [27] who extended Kiselman's radius of convergence problem to open sets U in E and obtained a close solution to it.

We should draw attention to the fact that the radius of convergence problem is a purely infinite dimensional one where $R_f \leq d_U$. This is simply because, for $E = \mathbb{C}^n$ every $f \in H(U)$ (U open in E) satisfies $R_f \geq d_U$. In particular $R_f = \infty$ if $U = \mathbb{C}^n$.

It is worth to note that condition (1) implies the pseudoconvexity of U provided that E is a PB-space.

Hence the problem of finding holomorphic function with prescribed radius of convergence is closely related to the Levi problem which can be formulated as follows: **Given a pseudoconvex domain U , does there exist a function $f \in H(U)$ with $R_f \leq d_U$, i.e. with U being the domain of existence of f .**

As a matter of fact this close relation encouraged us to begin to tackle the radius of convergence problem in non-locally convex spaces and to obtain in [2] the following close solution to it.

Theorem 1.2: Let E be an infinite-dimensional complex metric vector space with invariant metric d and with finite dimensional Schauder decomposition (π_n) such that $x \mapsto \log d(x, 0)$ is plurisubharmonic. Let $R: U \rightarrow \mathbb{R}_+$ be defined on a domain $U \subseteq E$ with $-\log R$ is plurisubharmonic and $R \leq d_U$. Then there exists a holomorphic function $f \in H(U)$ with radius of convergence $R_f \leq R$.

(For normed spaces the result is due to Schottenloher [27]).

Now we give the solution of the Levi problem in PB-spaces as consequence of the above theorem.

Theorem 1.3: (Levi problem in PB-spaces). Let E be an infinite-dimensional PB-space with f.d. decomposition (π_n) such that $x \mapsto \log d(x, 0)$ is plurisubharmonic in E . Then every pseudoconvex domain U in (or spread over) E is a domain of holomorphy.

For the proof we take $R = d_U$. Since U is pseudoconvex, $-\log R$ is plurisubharmonic. Applying theorem 1 we find a holomorphic function $f \in H(U)$ with $R_f \leq R = d_U$. This implies that U is a domain of existence of f and consequently a domain of holomorphy.

Application 1 [2]. (Levi problem in locally bounded spaces). Let E be a locally bounded space with f.d. decomposition. Then every pseudoconvex domain U in (or spread over) E is a domain of holomorphy.

The topology of any locally bounded space can be defined by a p -homogeneous norm $\|\cdot\|_p$ for some p ($0 < p \leq 1$). We can apply Theorem 1.2 noticing that E with $\|\cdot\|_p$ is a PB-space and $x \mapsto \log \|x\|_p$ is plurisubharmonic by Lelong [21].

If in addition to the hypotheses of Theorem 1.1, R is locally Lipschitz in U then we get an $f \in H(U)$ with $R_f \leq R$ and a lower bound for R_f as the following theorem shows (see the author [2]):

Theorem 1.4. Assume that E , U and R satisfying the hypotheses of theorem 1.2. If in addition $|R(x) - R(y)| \leq c d(x, y)$ for $x, y \in \Omega$, $d(x, y) < d_U(x)$, then there exists $f \in H(U)$ and a constant $k = k(c) > 0$ depending on c with $\phi(k)R \leq R_f \leq R$, where $\phi: R \rightarrow R$ is a continuous function with $\phi(0) = 0$, $\phi(t) > 0$ for $t > 0$; and $\phi(k) = k$ when E has a monotone Schauder decomposition.

The following result is a useful application of the above theorem.

Application 2 [2]. The spaces $\ell_{(p_n)} = \{x = (x_n); \sum |x_n|^{p_n} < \infty\}$, ($0 < p_n \leq 1$) with the metric $d(x, 0) = \sum_{n=1}^{\infty} |x_n - y_n|^{p_n}$ has a Schauder basis. It was shown in [24] that the linear and topological properties of $\ell_{(p_n)}$ depends on the sequence (p_n) we choose.

For example $\mathcal{L}_{(p_n)}$ is locally bounded if $p_n \rightarrow 0$ and $\mathcal{L}_{(p_n)}$ is not locally pseudoconvex if $p_n \rightarrow 0$, Rolewicz [22, p.153]

Now, regarding the Levi problem, the solution does not depend on the choice of (p_n) . In fact, the difference in linear and topological properties will not affect these holomorphic property, that is, all $\mathcal{L}_{(p_n)}$ are Levi spaces by theorem 1.3.

Moreover, all such spaces have the interesting holomorphic property that the radius of convergence problem has an exact solution. That is, given a function R with properties (1) and (2) there exists a holomorphic function f with $R_f = R$, See the author [2, prop. 3.6.1].

In the following we give another application of the result obtained for the radius of convergence problem of theorem 1.2

Application [3]. (Extension of Aron's theorem). We note that the function $f(x) = \sum_{k=0}^{\infty} x_k^k$ is entire and has $R_f(x) = 1$ at every $x \in \mathcal{L}_p (1 > p > 0)$, see the author [2].

Now we generalise Aron's theorem for normed spaces to PB-spaces E with Schauder bases. That is we show that it can even happen that $\inf_{x \in A} R_f(x) = 0$ for every closed non-compact set A in E .

In fact, the problem of constructing holomorphic functions with this or other similar properties is sometimes reduced to the much simpler problem of constructing plurisubharmonic functions with a certain growth by means of theorem 1.2.

To illustrate how this theorem implies Aron's theorem cited above we just take $\varepsilon > 0$ and define

$$\phi(x) = \sup_n n(|x_n| - \varepsilon)^+, \quad x = (x_n) \in c_0$$

Then ϕ is convex, hence plurisubharmonic, on every ℓ^p , $0 < p < \infty$, and still unbounded in every ball $B(x, r)$ of radius $r \geq \varepsilon$. An entire function $f \in H(\ell^p)$ with radius of convergence $R_f \leq e^{-\phi}$ has the desired properties. This can also be carried out for sequence PB-spaces.

Using the results of the above theorem we show the richness of the class $H(U)$ on a pseudoconvex open set U in a PB-spaces.

Application 3[2]. (Richness of $H(U)$). Let E , U and R be as in Theorem 1.2. Then the set $T_1(\rho) = \{f \in H(U), R_f < R\}$ is sequentially dense in $(H(U), \tau_0)$. Moreover, if R in addition to this is locally lipschitz i.e. satisfies the hypotheses of Theorem 1.4, then $T_2(\rho) = \{f \in H(U); KR < R_f < R\}$ is sequentially dense in $(H(U), \tau_0)$, where τ_0 is the compact open topology on $H(U)$.

As a counter example to the Levi problem in non-locally convex space we give

Example 4. The space $L^p = \{f; \int_0^1 |f|^p d\mu < \infty\}$ of measurable functions metrized by the metric $d(f, g) = \int_0^1 |f - g|^p d\mu$, ($0 < p < 1$) has the dual space $(L^p)' = \{0\}$. Hence a domain $U \neq L^p$ (pseudoconvex or not) can never be a domain of holomorphy. (for a counter example of a locally convex spaces see Josefson [16]).

2. The Levi problem in F-spaces and Gruman-Kiselman approach

We shall focus our attention on the basic approach given by Gruman and Kiselman[14] which was used to solve the Levi problem in Banach spaces with bases. This represents the common method that has been used by several authors [cf. 3,8,13,14] to solve this problem in different classes of locally convex spaces. The technique is to construct a holomorphic function on a pseudoconvex domain U which is unbounded on sufficiently many subsequences of a sequence (x_n) in U with the property that the accumulation points of (x_n) are dense in ∂U .

We have used this method to solve the Levi problem for domains spread over the following:

- 1) metrizable spaces with f.d. decomposition and
- 2) locally pseudoconvex F-spaces with the b.a.p.

Although the radius of convergence technique (which has been used in section 1 is stronger than this one, the geometry of the non-locally convex spaces may not enable us to use this technique for general metric vector spaces.

The following theorem gives the solution of the Levi problem for a big class of F-spaces (see the author [3]).

Theorem 2.1. (Levi problem for F-spaces). Let E be an infinite dimensional metrizable space with f.d. decomposition (π_n) . Then every pseudoconvex domain in E is a domain of holomorphy. That is, E is a Levi space.

The method which we have used in [3] to prove the above theorem shows that we can use the same cover for U that has been used by Gruman and Kiselman [14] to establish our proof for F -spaces.

For locally bounded F -spaces E , this is obvious since E is defined by a p -norm $\|\cdot\|_p$ and $x \mapsto \log \|x\|_p$ is plurisubharmonic, see Lelong [21, p. 153]. Also $x \mapsto -\log d_U(x)$ is plurisubharmonic in U whenever U is pseudoconvex, see the author [2].

The following example is of special interest since a separable Banach space is isomorphic to a quotient space of ℓ^+_O .

Example 5. Let $\ell^+_{p_O} = \bigcap_{p \geq p_O} \ell^+_p$ ($1 \geq p_O > 0$). Then $\ell^+_{p_O}$ is a locally pseudoconvex F -space with respect to the F -norm $\|x\| = \sup_{p \geq p_O} \|x\|_p$. It is neither locally bounded nor locally convex.

A sequence converges to zero in E iff it converges to zero in ℓ^+_p for all $p \geq p_O$. It is proved that every convex closed norm-bounded convex subset of $\ell^+_{p_O}$ is compact. Hence no infinite-dimensional subspace is isomorphic to a normed space, see Shapiro [24].

Now having the fact that $\ell^+_{p_O}$ are F -spaces with bases we obtain that every pseudoconvex domain U in or spread over $\ell^+_{p_O}$ is a domain of holomorphy as a consequence of theorem 2.1.

To solve the Levi problem in a locally pseudoconvex F -space with the b.a.p. E , we should investigate whether E is isomorphic to a complemented subspace of a metrizable space with basis so that theorem 2.1 can be applied. The following theorem will establish this fact. (see the author [3]).

Theorem 2.2. Let E be a locally pseudoconvex F -space. Then E has the b.a.p. if and only if E is a complemented subspace of a metrizable locally pseudoconvex space with basis.

In fact, the above theorem represents a generalization of Plélczynski's theorem in locally convex Fréchet spaces. Making use of the above two theorems we get

Theorem 2.3. (Levi problem for locally pseudoconvex F-spaces). Every pseudoconvex domain U in (or spread over) a locally pseudoconvex F-space E with the b.a.p is a domain of holomorphy.

3. The Levi problem in locally pseudoconvex spaces and surjective limit approach.

In this section we are going to deal with the preservation of the property of being a Levi space under surjective limits. Our setting here is the **Locally pseudoconvex spaces** (Lps) which may not be metrizable or locally convex. This includes a special class, the p -Banach spaces ($1 \leq p < \infty$), see the author [4].

We shall first describe some basic properties of surjective limits which can be obtained conveniently as for locally convex spaces.

A Lps E is a **surjective limit** of Lps $(E_i)_{i \in A}$ (and we write $E = \varprojlim_{i \in A} E_i$) if there exists a continuous linear mapping π_i from E onto E_i , for each $i \in A$ and the inverse images of the neighborhoods of 0 in E_i , as i ranges over A , form a basis for the neighborhood system at 0 in E . That is, for each neighborhood W of 0 in E , there exists an $i \in A$ and a neighborhood V of 0 in E_i such that $\pi_i^{-1}(V) \subseteq W$. E is called **open surjective limit** if π_i is open map for each $i \in A$.

The surjective limit approach was used by Hirschowitz [15] in studying the Cartesian product of the complex planes, and by Dineen [9,10] in characterising the collection of locally convex spaces (l.c.s) in which certain holomorphic properties are true.

The following example has useful consequences.

Example 7. Let $CP(E)$ denote the set of all continuous pseudonorms (i.e. p_α -seminorms for some $p_\alpha (0 < p_\alpha \leq 1)$) on a Lps E . For each $q \in CP(E)$, let E_q denote the vector space E equipped with the topology generated by q and let π_q denote the canonical map from E onto $E_q/q^{-1}(0)$. Then E has the surjective representation $(E_q/q^{-1}(0), \pi_q)$ which will be called pseudonorm surjective representation of E .

In what follows we introduce a big collection of Lps spaces which are closed under the operation of surjective limit.

Lemma 3.1. The collection of locally pseudoconvex spaces $(E_i)_{i \in A}$ with the approximation property (a.p.) is closed under the operation of surjective limit; that is, $\varinjlim_{i \in A} E_i$ has the a.p. if each $E_i, i \in A$, has the a.p.

For the proof notice that the a.p. of a topological vector space E does not depend on whether or not E is locally convex. But it does depend on the existence of a family $(\pi_i)_{i \in A}$ of finite-rank continuous linear functionals $\pi_i : E \rightarrow E$ which can be used to approximate the identity mapping I . So we may follow here Dineen's method [9, Example 2.3] of a locally convex space.

The following proposition will be helpful to reduce the study of the Levi problem for Lps to spaces which have continuous p_α -norms ($0 < p_\alpha \leq 1$).

Proposition 3.2. Every Hausdorff Lps E with f.d. decomposition (π_i) is a surjective limit of spaces E_i , $i \in A$, where E_i has a continuous p_i -norm and f.d. decomposition.

This can be established in a manner similar to that of Dineen [9, Example 2.4] taking into account Example 7.

The next result gives a sufficient condition on a pseudoconvex domain U of a Lps E so that $U = \pi^{-1}(\pi(U))$, where $\pi: E \rightarrow E/q^{-1}(0)$ is the quotient map and q is a continuous p -seminorm on E .

Lemma 3.3. Let U be a pseudoconvex domain of a Lps E . Let q be a continuous p -semi-norm on E ($0 < p < 1$) such that $B_q(0, \delta) \subseteq U$ for $\delta > 0$. Then

$$U = U + \{y; q(y) = 0\}$$

The proof of the above lemma is a finite dimensional one which does not depend on whether or not E is Lps. So we can follow Dineen's method [10, lemma 1.1] taking into account the result of Hirschowitz [15].

The following lemma is of special interest when dealing with the solution of the Levi problem. The proof is analogous to that given by Dineen [10, lemma 1.8] using lemma 3.3 instead of the corresponding result in normed spaces.

Lemma 3.4. Let U be an open pseudoconvex subset of open surjective limit $\lim_{\leftarrow} (E_i, \pi_i)$ of locally pseudoconvex spaces E_i , $i \in A$. Then U is π_i -open for some $i \in A$.

(i.e., for every $x \in U$, there exists V_x open in E_i such that $x \in \pi_i^{-1}(V_x) \subseteq U$).

We are now in a position to introduce a basic theorem, which will enable us to get the solution of the Levi problem for Lps spaces.

Theorem 3.5. The collection of Hausdorff Lps spaces with the b.a.p in which pseudoconvex domains are domains of holomorphy is closed under open surjective limit. That is, the open surjective limit of the Levi spaces with the b.a.p. is a Levi space.

To establish this, we assume that U is a pseudoconvex domain of the surjective limit $E = \varinjlim_{i \in A} E_i$. Then there exists an $i \in A$ such that $U = \pi_i^{-1}(\pi_i(U))$ with $\pi_i(U)$ pseudoconvex domain (see lemma 3.4).

By assumption, $\pi_i(U)$ is a domain of holomorphy. Now if there exist open connected sets U_1, U_2 in E such that $U_2 \subset U$, $U \cap U_1 \supset U_2$ and for each $f \in H(U)$, there is an $f_1 \in H(U_1)$ with $f|_{U_2} = f_1|_{U_2}$. Then, as a consequence of π_i being open we obtain $\pi_i(U_1) \subseteq \pi_i(U)$. Note that $\pi_i(U)$ is a domain of holomorphy. Hence $U_1 \subseteq \pi_i^{-1}(\pi_i(U)) = U$ and consequently U is a domain of holomorphy. This completes the proof of the theorem.

We are now in a position to give the solution of the Levi problem in Lps having the b.a.p. (The case of l.c.s was considered by Dineen [9,10] and Schottenloher [25]).

Theorem 3.6 (Levi problem in Lps). Let E be a Hausdorff Lps with the b.a.p. Then every pseudoconvex U in (or spread over) E is a domain of holomorphy.

Note that every Lps E can be expressed as a surjective limit of p_α -normed spaces with continuous p_α -norms q_α (see Example 7). Hence applying theorem 3.5 and the solution of the Levi problem for p -normed spaces with b.a.p. given by the author [3] (see theorem 2.3) we obtain the required result.

The following is an example of a non-metrizable Lps.

Example 8[7]. (The inductive space $\bigcup_{1 \leq p < \infty} \ell_p$)

We define the q -topology on $\bigcup_{1 \leq p < \infty} \ell_p$ to be the strongest vector topology such that each injection $i_p: \ell_p \rightarrow \bigcup_{1 \leq p < \infty} \ell_p$ is continuous. The space $\bigcup_{1 \leq p < \infty} \ell_p$ with this q -topology is complete separable non-locally convex Lps and has the unit vectors (e_n) as its symmetric Schauder basis. A sequence converges in $\bigcup_{1 \leq p < \infty} \ell_p$ iff it is contained in and converges in some ℓ_p . A set is compact in $\bigcup_{1 \leq p < \infty} \ell_p$ iff the set is contained and compact in some ℓ_p ; no closed infinite dimensional subspace of $\bigcup_{1 \leq p < \infty} \ell_p$ is contained in ℓ_p ; and no infinite dimensional subspace of $\bigcup_{1 \leq p < \infty} \ell_p$ is metrizable. Hence $\bigcup_{1 \leq p < \infty} \ell_p$ itself is not metrizable, see Stiles [24].

Since E is Lps with Schauder basis, theorem 3.6 implies that E is a Levi space.

Acknowledgements

This work was partially supported by grant No. Math/1406/30 from the Research Center of the Faculty of Science, King Saud University.

References

- [1] R. Aron, Entire function of unbounded type on Banach space. Boll. Un. Mat. Ital. (4)9, 28-31 (1974).
- [2] A. Bayoumi, The Levi problem and the radius of convergence of holomorphic functions on metric vector spaces, Lecture Notes in Math. 843, pp. 9-32, Springer Verlag (1981).
- [3] A. Bayoumi, The Levi problem for Domains spread over locally pseudoconvex Fréchet spaces with the bounded approximation property: J. complex variables Vol. 10, pp. 141-152 (1988).
- [4] A. Bayoumi, The Levi problem in non-locally convex separable topological vector spaces. Math. Scandinavica 67 (1990).
- [5] A. Bayoumi. Bounding subsets of some metric vector spaces. Arkiv. for Matematik 18(1), 1980 p. 13-17, Sweden.
- [6] A. Bayoumi. On Holomorphic Hahn-Banach extension theorem and properties of bounding and weakly bounding sets in some metric vector spaces. Port. Math. Vol.(46) 1989.
- [7] A. Bayoumi. The theory of bounding subsets of topological vector spaces without convexity condition. Port. Math. Vol.(47), (1990) 25-42.
- [8] J.F. Colombeau and J. Mujica, The Levi problem in nuclear Silva spaces. Arkiv for Mat. 18(1), p. 13-17 (1980).
- [9] S. Dineen, Surjective limits of locally convex spaces and their application to infinite dimensional holomorphy. Bull. Soc. Math. France 103, 441- 509 (1975).
- [10] S. Dineen, Holomorphic functions on locally convex spaces. II-pseudoconvex domains, Ann. Inst. Fourier. Grenoble 23, (1973) 155-185.

- [11] S. Dineen, P. Noveraz et M. Schotteuloher, Le problème de Levi dans certains espaces vectoriels topologiques localement convexes. Bull. Soc. Math. France 104 (1976), 87-97.
- [12] S. Dineen, Complex Analysis in locally convex spaces. North-Holland, 57 (1983).
- [13] L. Gruman, The Levi problem in certain infinite dim. vector spaces. Illinois J. Math. 18, p. 20-26 (1972).
- [14] L. Gruman & C.O. Kiselman, Le problème de Levi dans les espaces de Banach à base. C.R. Acad. Sc. Paris A. 274, 1290-1299 (1972).
- [15] A. Hirschowitz, Remarques sur les ouverts d'holomorphie d'un produit dénombrable de droites. Ann. Inst. Fourier. Grenoble 19,1 (1969), 219-229.
- [16] B. Josefson, A counter example in the Levi problem. In: proceedings on infinite dimensional holomorphy. Lecture Notes in Mathematics 364, 168-177, springer verlag (1974).
- [17] C.O. Kiselman, On the radius of convergence of an entire function in normed space. Ann. Polon. Math. 33 (1976), 39-53.
- [18] C.O. Kiselman, Geometric aspects of the theory of bounds for entire functions in normed spaces. In: Infinite dimensional holomorphy and applications. Ed.M.C. Matos. North-Holland 1977.
- [19] C.O. Kiselman, Constructions des fonctions entières à rayon de convergence donné. Lecture Notes in Mathematics 579, 246-253. springer-verlag (1977).
- [20] L. Nachbin, Recent developments in infinite dimensional holomorphy B.A.M.S., 79 (1973), p. 625-640.

- [21] P. Lelong, Recent results on analytic mapping and plurisubharmonic functions in topological vector spaces. Lecture Notes Mathematics 185 (1971).
- [22] S. Rolewicz, Metric Linear spaces. Instytut Matematyczny polskiej Akademii Nauk, Monografie Matematyczne (1972).
- [23] J. Stiles, On the inductive limit of U_p ($0 < p < 1$). Studia Math. LVI, 1 (1976).
- [24] J. Shapiro, On convexity and compactness in F-spaces with bases. Indiana University Math. Journal. Vol. 21. No. 12 (1971).
- [25] M. Schattenloher, The Levi problem for domains spread over locally convex spaces with a finite dimensional Schauder decomposition. Ann. Inst. Fourier 26 (1976), 207-377.
- [26] M. Schottenloher, Richness of the class of holomorphic functions on an infinite dimensional space. Functional Analysis, Results and surveys. Conference in Paderborn, North-Holland 1976.
- [27] M. Schottenloher, Holomorphe Funktionen auf Gebieten Über Barach raumen Zu Vorgegebenen konvergenzstrahlungen. Manuscripta Math. 21 (1977), 315-327.

Established in 1978 by Prof. R. M. Santilli at Harvard University Hadronic Journal and Algebras, Groups and Geometries have been regularly published since 1978 without publication charges and are among the few remaining independent refereed journals.

This Journal publishes advances research papers and Ph. D. theses in any field of mathematics without publication charges.

For subscription, format and any other information please visit the website
<http://www.hadronicpress.com>

HADRONIC PRESS INC.
35246 U. S. 19 North Suite 215
Palm Harbor, FL 34684, U.S.A.
<http://www.hadronicpress.com>
Email: info@hadronicpress.com
Phone: +1-727-946-0427

ISSN: 0741 - 9937



HADRONIC PRESS, INC.